Incompatibility measures in multiparameter quantum estimation under hierarchical quantum measurements

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The incompatibility of the measurements constrains the achievable precisions in multiparameter quantum estimation. Understanding the tradeoff induced by such incompatibility is a central topic in quantum metrology. Here we provide an approach to study the incompatibility under general p-local measurements, which are the measurements that can be performed collectively on at most p copies of quantum states. We demonstrate the power of the approach by presenting a hierarchy of analytical bounds on the tradeoff among the precisions of different parameters. These bounds lead to a necessary condition for the saturation of the quantum Cramér-Rao bound under p-local measurements, which recovers the partial commutative condition at $p = 1$ and the weak commutative condition at $p = \infty$. As a further demonstration of the power of the framework, we present another set of tradeoff relations with the right logarithmic operators.

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I. INTRODUCTION

By utilizing quantum mechanical effects, such as superposition and entanglement, quantum metrology can achieve better precision limit than classical metrology. There is now a good understanding on the local precision limit for single-parameter quantum estimation, where the precision limit can be quantified by the single-parameter quantum Cramér-Rao bound [1–14]. Practical applications, however, typically involve multiple parameters, for which the precision limits are much less understood [15–42]. Due to the incompatibility of the optimal measurements for different parameters, the multiparameter quantum Cramér-Rao bound is in general not achievable. Tradeoffs among the precisions of different parameters have to be made. Quantifying such tradeoff is now one of the main subjects in quantum metrology [28–50].

The incompatibility of the measurements is rooted in the prohibition of simultaneous measurement of noncommutative observables, which is one of the defining features of quantum mechanics. Previous studies on the incompatibility mostly focus on the extreme cases: either the measurement is separable or can be performed collectively on infinite copies of quantum states. When the measurements can be performed on infinite number of identical copies of quantum states, the Holevo bound quantifies the achievable precision [2,46–48]. Except for few special cases [49,50], the Holevo bound in general can only be evaluated numerically [27]. In practice the measurements typically can only be performed collectively on a finite number of quantum states, under which the Holevo bound is also not achievable in general. In the case of two parameters, Nagaoka provided a bound under the separable measurements which is tighter than the Holevo bound [51,52]. Conlon *et al.* recently generalized the Nagaoka bound to more than two parameters, which in general requires numerical optimization [53]. Gill-Massar bound provided an analytical measure on the tradeoff induced by the incompatibility of the separable measurements [28]. For collective measurements on at most two copies of quantum states, Zhu and Hayashi have obtained a tradeoff relation for completely unknown states [25]. However, the incompatibilities under general p-local measurements, which are the measurements that can be performed collectively on at most p copies of quantum states, are little understood.

Here we provide a framework to study the precision under general p-local measurements. This approach leads to multiparameter precision bounds which include the Holevo bound and the Nagaoka bound as special cases. We also provide a systematic way to generate hierarchical analytical tradeoff relations under general p-local measurements. The obtained tradeoff relations provide a necessary condition for the saturation of the multiparameter quantum Cramér-Rao bound under p-local measurements, which recovers the partial commutative condition [23] at $p = 1$ and the weak commutative condition at $p = \infty$. Our study thus not only provides a framework that can generate analytical bounds on the tradeoff under general p-local measurements, but also provides a coherent picture for the existing results on the extreme cases.

The article is organized as following: in Sec. II we introduce the notations and list the main results; in Sec. III we present analytical tradeoff relations for pure states; in Sec. IV we provide multiparameter precision bounds for mixed states and use it to derive analytical tradeoff relations for mixed states. The tradeoff relations lead to a necessary condition for the saturation of the quantum Cramér-Rao bound (QCRB) and we show how it reduces to the partial commutative condition at $p = 1$ and the weak commutative condition at $p \to \infty$; in Sec. V we demonstrate the versatility of the approach by

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presenting another set of tradeoff relations in terms of the right logarithmic derivative; in Sec. VI some examples are presented and Sec. VII concludes. This paper is an extended version of the companion paper [54] which focuses on the information geometry under hierarchical quantum measurements.

II. PRECISION LIMIT IN QUANTUM METROLOGY

We first introduce the notations and terminologies that are used in this article and list the main results.

For the single-parameter quantum estimation, given a parametrized state ρ_x, with x as the parameter to be estimated, by performing a positive-operator-valued measurement (POVM), denoted as $\{M_\alpha\}$, on the state, we can get the measurement result α with a probability $p(\alpha|x) = \mathrm{Tr}(\rho_x M_\alpha)$. The variance of any locally unbiased estimator $\hat{x}$ is then lower bounded by the Cramér-Rao bound [55,56] as $\delta\hat{x}^2 \geqslant \frac{1}{\nu F_C}$, here $\delta\hat{x}^2 = E[\hat{x} - x]^2$ is the variance of the unbiased estimator, $F_C = \int_\alpha \frac{1}{p(\alpha|x)}(\frac{\partial_x p(\alpha|x)}{\partial x})^2 d\alpha$ is the Fisher information [56], ν is the number of repetitions of the procedure, which is assumed to be asymptotically large. By optimizing the POVM, we get the quantum Cramér-Rao bound (QCRB) [1,2]

$$\delta\hat{x}^2 \geqslant \frac{1}{\nu F_C} \geqslant \frac{1}{\nu F_Q}, \tag{1}$$

where F_Q is the quantum Fisher information (QFI), which is the maximization of the Fisher information over all POVM [1,2]. The QFI can be computed directly from the quantum state as $F_Q = \mathrm{Tr}(\rho_x L^2)$, here L is the symmetric logarithmic operator (SLD) which is implicitly defined via the equation $\frac{\partial \rho_x}{\partial x} = \frac{1}{2}(L\rho_x + \rho_x L)$. For single-parameter estimation, the QCRB can always be saturated with the POVM performed separately on each copy of the state. One POVM that saturates the single-parameter QCRB is the projective measurement on the eigenvectors of the SLD.

For multiparameter quantum estimation, where $x = (x_1, \dots, x_n)$ with $n \geqslant 2$, the quantum Fisher information becomes the quantum Fisher information matrix with the jkth entry given by

$$(F_Q)_{jk} = \mathrm{Tr}\left(\rho_x \frac{L_j L_k + L_k L_j}{2}\right), \tag{2}$$

where L_q is the SLD corresponding to the parameter x_q, which satisfies $\partial_{x_q}\rho_x = \frac{1}{2}(\rho_x L_q + L_q \rho_x)$, $\forall\, q \in \{1, \dots, n\}$. The multiparameter quantum Cramér-Rao bound is given by

$$\mathrm{Cov}(\hat{x}) \geqslant \frac{1}{\nu} F_Q^{-1}, \tag{3}$$

where $\mathrm{Cov}(\hat{x})$ is the covariance matrix for locally unbiased estimators, $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$, with the jkth entry given by $\mathrm{Cov}(\hat{x})_{jk} = E[(\hat{x}_j - x_j)(\hat{x}_k - x_k)]$, ν is the number of copies of quantum states. In this article, we assume F_Q is nonsingular so F_Q^{-1} exists, in which case $\mathrm{Cov}(\hat{x}) \geqslant \frac{1}{\nu} F_Q^{-1} > 0$ is also nonsingular.

Different from the single-parameter quantum estimation, the multiparameter quantum Cramér-Rao bound is in general not saturable. This is due to the incompatibility of the optimal measurements for different parameters. Such incompatibility is rooted in the prohibition of simultaneous measurement of noncommutative observables and its manifested effect in multiparameter estimation is the tradeoff on the precision limits for the estimation of different parameters.

We can quantify the incompatibility through either $\frac{1}{\nu}\mathrm{Tr}[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})]$ [29] or $\nu\,\mathrm{Tr}[F_Q \mathrm{Cov}(\hat{x})]$ [20,29,30], which measures how close $\mathrm{Cov}(\hat{x})$ is to $\frac{1}{\nu}F_Q^{-1}$. These two quantities are roughly reciprocal to each another. Compared to the other quantities, such as $\|\nu\,\mathrm{Cov}(\hat{x}) - F_Q^{-1}\|$ or $\|\frac{1}{\nu}\mathrm{Cov}^{-1}(\hat{x}) - F_Q\|$, $\frac{1}{\nu}\mathrm{Tr}[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})]$ and $\nu\,\mathrm{Tr}[F_Q\mathrm{Cov}(\hat{x})]$ both have the advantage of being invariant under reparametrization. In this article we will use $\Gamma = \frac{1}{\nu}\mathrm{Tr}[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})]$ as the measure. When the QCRB is saturable, $\mathrm{Cov}(\hat{x}) = \frac{1}{\nu}F_Q^{-1}$, $\Gamma = \mathrm{Tr}(I_n) = n$, here I_n denotes the $n \times n$ identity matrix. This is the maximal value Γ can achieve. When the QCRB is not saturable we have $\Gamma < n$. The gap between n and Γ quantifies the incompatibility. We will denote the measure under the p-local measurement as $\Gamma_p = \frac{1}{\nu}\mathrm{Tr}[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})]$ with $\mathrm{Cov}(\hat{x})$ as the covariance matrix achieved under the optimal p-local measurement. The gap between n and Γ_p decreases with p since the measurements become less restrictive when p increases, we thus have $\Gamma_1 \leqslant \Gamma_2 \leqslant \cdots \leqslant \Gamma_\infty$. For pure states, however, we have $\Gamma_1 = \Gamma_2 = \cdots = \Gamma_\infty$ since for pure states the optimal measurement can be taken as the 1-local measurement [15].

The existing results on the incompatibility are mostly on the extreme cases with either $p = \infty$ or $p = 1, 2$. For $p = \infty$, the precision limit can be characterized by the Holevo bound [2], which is given by $\nu\,\mathrm{Tr}[W\,\mathrm{Cov}(\hat{x})] \geqslant \min_{\{X_j\}}\{\mathrm{Tr}[W\,\mathrm{Re}Z(X)] + \|\sqrt{W}\,\mathrm{Im}Z(X)\sqrt{W}\|_1\}$, where $W \geqslant 0$ is a weighted matrix, $Z(X)$ is a matrix with its jkth entry given by $Z(X)_{jk} = \mathrm{Tr}(\rho_x X_j X_k)$, where $\{X_1, \dots, X_n\}$ is a set of Hermitian operators that satisfy the local unbiased condition $\mathrm{Tr}(\rho_x X_j) = 0$ for any $j \in \{1, \dots, n\}$ and $\mathrm{Tr}(\partial_{x_k}\rho_x X_j) = \delta_k^j$ with δ_k^j as the Kronecker delta, $\delta_k^j = 1$ when $k = j$ and $\delta_k^j = 0$ when $k \neq j$, $\mathrm{Re}Z(x) = \frac{Z(x)+Z^T(x)}{2}$ is the real part of $Z(x)$, $\mathrm{Im}Z(x) = \frac{Z(x)-Z^T(x)}{2i}$ is the imaginary part. The Holevo bound can only be evaluated numerically in general [27]. For pure states, the Holevo bound can be saturated by 1-local measurements [15]. For mixed states, the saturation of the Holevo bound in general requires collective measurements on infinite copies of the state.

A necessary and sufficient condition for the Holevo bound to coincide with the QCRB is $\mathrm{Tr}(\rho_x[L_j, L_k]) = 0$ for any $j, k \in \{1, \dots, n\}$. This is called the weak commutative condition [15]. When the weak commutative condition holds, there exist collective measurements on infinite copies of quantum states under which the QCRB is saturated and $\Gamma_\infty = n$.

As the Holevo bound corresponds to the minimal value upon all choice of $\{X_j\}$, by making a particular choice of $\{X_j\}$ as $X_j = \sum_k (F_Q^{-1})_{jk} L_k$ and $W = F_Q$, we have [30]

$$\nu\mathrm{Tr}[F_Q\mathrm{Cov}(\hat{x})] \leqslant n + \left\|F_Q^{-\frac{1}{2}} F_{\mathrm{Im}} F_Q^{-\frac{1}{2}}\right\|_1 \leqslant 2n, \tag{4}$$

where F_{Im} is a matrix with the jkth entry given by $(F_{\mathrm{Im}})_{jk} = \frac{1}{2i}\mathrm{Tr}(\rho_x[L_j, L_k])$. The last inequality is obtained from the fact that $F_Q + iF_{\mathrm{Im}} \geqslant 0$, which leads to $\|F_Q^{-\frac{1}{2}} F_{\mathrm{Im}} F_Q^{-\frac{1}{2}}\|_1 \leqslant \mathrm{Tr}(F_Q^{-\frac{1}{2}} F_Q F_Q^{-\frac{1}{2}}) = n$. Through the Cauchy-Schwarz inequality,

$$\mathrm{Tr}[F_Q\mathrm{Cov}(\hat{x})]\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] \geqslant \big|\mathrm{Tr}\big[F_Q^{\frac{1}{2}}\mathrm{Cov}^{\frac{1}{2}}(\hat{x})\mathrm{Cov}^{-\frac{1}{2}}(\hat{x})F_Q^{-\frac{1}{2}}\big]\big|^2 = n^2, \tag{5}$$

this leads to a lower bound on Γ_∞ as [20]

$$\Gamma_\infty = \frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] \geqslant \frac{n^2}{n + \big\|F_Q^{-\frac{1}{2}}F_{\mathrm{Im}}F_Q^{-\frac{1}{2}}\big\|_1}. \tag{6}$$

We note that the lower bound on Γ_∞ is not sufficient to decide the incompatibility of the measurements at $p=\infty$ as it can not tell whether Γ_∞ can reach n and furthermore how close Γ_∞ is to n. The upper bound is more informative in this sense. As if there exists an upper bound which is less than n, we can tell for sure that the measurements are incompatible, and furthermore the gap between n and the upper bound provides a measure on the incompatibility. To our knowledge, except the trivial bound $\Gamma_\infty \leqslant n$, there were no analytical upper bounds on Γ_∞.

For the other extreme case with $p=1$, Nagaoka provided a bound on the precision limit in the case of two parameters ($n=2$) [51,52], as

$$\nu\,\mathrm{Tr}[\mathrm{Cov}(\hat{x})] \geqslant \min_{\{X_1,X_2\}} \mathrm{Tr}\big(\rho_x X_1^2\big) + \mathrm{Tr}\big(\rho_x X_2^2\big) + \|\sqrt{\rho_x}[X_1,X_2]\sqrt{\rho_x}\|_1, \tag{7}$$

where $\{X_1,X_2\}$ are two Hermitian operators satisfying the locally unbiased condition. The Nagaoka bound in general can only be evaluated numerically and is tighter than the Holevo bound. Recently, the Nagaoka bound has been generalized to n parameters which also requires numerical evaluation [53].

Gill and Massar provided an analytical upper bound on Γ_1 as [28]

$$\Gamma_1 \leqslant d-1, \tag{8}$$

where d is the dimension of the Hilbert space for a single ρ_x. The Gill-Massar bound is nontrivial only when $n > d-1$. Recent studies have also obtained some tradeoff relations with the Ozawa's uncertainty relation for pairs of parameters [31].

A necessary condition for the saturation of the QCRB under 1-local measurements is the partial commutative condition [23], which requires all SLDs commute on the support of ρ_x. Specifically if we write ρ_x in the eigenvalue decomposition as $\rho_x = \sum_1^m \lambda_i|\Psi_i\rangle\langle\Psi_i|$ with $\lambda_i > 0$, the partial commutative condition is $\langle\Psi_r|[L_j,L_k]|\Psi_s\rangle = 0$ for any $j,k \in \{1,\ldots,n\}$, and any $r,s\in\{1,\ldots,m\}$. The connection between the partial commutative condition and the weak commutative condition remained open [23].

For $p=2$, Zhu and Hayashi provided an upper bound on Γ_2 as

$$\Gamma_2 \leqslant \frac{3}{2}(d-1), \tag{9}$$

which is nontrivial only when $n > \frac{3}{2}(d-1)$.

For general p, the incompatibility is little understood. In this article, we provide a framework to study the incompatibility under general p-local measurements. This framework provides precision bounds that include the Holevo bound and the Nagaoka bound as special cases, and leads to nontrivial analytical upper bounds for general Γ_p. A necessary condition for the saturation of the QCRB can also be obtained, which recovers the partial commutative condition at $p=1$ and the weak commutative condition at $p\to\infty$. The multiparameter precision bounds are presented in Sec. IV A. Here we first list the analytical upper bounds and the necessary condition for the saturation of QCRB under general p-local measurements.

(1) For pure states, we have

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] \leqslant n - f(n)\big\|F_Q^{-\frac{1}{2}}F_{\mathrm{Im}}F_Q^{-\frac{1}{2}}\big\|_F^2, \tag{10}$$

where $\|\ldots\|_F$ is the Frobenius norm and n is the number of parameters, $f(n) = \max\{\frac{1}{4(n-1)}, \frac{n-2}{(n-1)^2}, \frac{1}{5}\}$ which can be equivalently written as

$$f(n) = \begin{cases} \frac{1}{4(n-1)} & \text{when} \quad n=2,\\ \frac{n-2}{(n-1)^2} & \text{when} \quad n=3 \text{ or } 4,\\ \frac{1}{5} & \text{when} \quad n\geqslant 5.\end{cases}$$

We note the bounds for pure states do not depend on p since for pure states $\Gamma_1 = \Gamma_2 = \cdots = \Gamma_\infty$.

(2) For mixed states under p-local measurements, we have

$$\Gamma_p \leqslant n - f(n)\left\|\frac{F_Q^{-\frac{1}{2}}\bar{\mathbf{F}}_{\mathrm{Im}p}F_Q^{-\frac{1}{2}}}{p}\right\|_F^2, \tag{11}$$

where $f(n) = \max\{\frac{1}{4(n-1)}, \frac{n-2}{(n-1)^2}, \frac{1}{5}\}$, $\bar{\mathbf{F}}_{\mathrm{Im}p}$ is the imaginary part of $\bar{\mathbf{F}} = \sum_q \bar{F}_{u_q}$ with each $\bar{F}_{u_q}$ equal to either F_{u_q} or $F_{u_q}^T$, where F_{u_q} is a $n\times n$ matrix with the jkth entry given by

$$\big(F_{u_q}\big)_{jk} = \langle u_q|\sqrt{\rho_x^{\otimes p}}L_{jp}L_{kp}\sqrt{\rho_x^{\otimes p}}|u_q\rangle, \tag{12}$$

L_{jp} is the SLD of $\rho_x^{\otimes p}$ corresponding to the parameter x_j, and $\{|u_q\rangle\}$ are any set of vectors in $H_d^{\otimes p}$ that satisfies $\sum_q |u_q\rangle\langle u_q| = I_{d^p}$ with I_{d^p} denote the $d^p\times d^p$ identity matrix.

(3) For mixed states under p-local measurements, we obtain another bound as

$$\Gamma_p \leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2, \tag{13}$$

where

$$(C_p)_{jk} = \frac{1}{2}\left\|\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{jp},\tilde{L}_{kp}]\sqrt{\rho_x^{\otimes p}}\right\|_1, \tag{14}$$

$\tilde{L}_{jp}$ is the SLD of $\rho_x^{\otimes p}$ under the reparametrization such that the quantum Fisher information matrix (QFIM) of ρ_x equals to the identity, specifically $\tilde{L}_{jp} = \sum_q (F_Q^{-\frac{1}{2}})_{jq}L_{qp}$ with L_{qp} as the SLD of $\rho_x^{\otimes p}$ corresponding to the original parameter x_q. We note that $\|\frac{C_p}{p}\|_F \geqslant \|\frac{F_Q^{-\frac{1}{2}}\bar{\mathbf{F}}_{\mathrm{Im}p}F_Q^{-\frac{1}{2}}}{p}\|_F$, this bound is thus tighter than the bound in Eq. (11) when $f(n) = \frac{1}{4(n-1)}$; however, it can be less tighter when $f(n) = \frac{n-2}{(n-1)^2}$ or $\frac{1}{5}$.

(4) From the above bound, we obtain a necessary condition for the saturation of the QCRB under p-local measurements, which is $\frac{C_p}{p} = 0$. For $p=1$, this reduces to the partial commutative condition. For $p\to\infty$, we prove that

$$\lim_{p\to\infty}\frac{(C_p)_{jk}}{p} = \frac{1}{2}|\mathrm{Tr}(\rho_x[\tilde{L}_j,\tilde{L}_k])|. \tag{15}$$

The condition $\frac{C_p}{p}=0$ thus reduces to the weak commutative condition $\mathrm{Tr}(\rho_x[\tilde{L}_j,\tilde{L}_k])=0, \forall\ j,k$, at $p\to\infty$. This clarifies the relation between the partial commutative condition and the weak commutative condition, which solves an open question [23].

(5) We provide another simpler bound for mixed states which can be calculated with operators only on a single ρ_x.

Given $\rho_x=\sum_{q=1}^m\lambda_q|\Psi_q\rangle\langle\Psi_q|$ with $\lambda_q>0$ in the eigenvalue decomposition, under p-local measurements we have

$$\Gamma_p\leqslant n-\frac{1}{4(n-1)}\left\|\frac{T_p}{p}\right\|_F^2, \tag{16}$$

where T_p is a $n\times n$ matrix with the jkth entry given by

$$(T_p)_{jk}=\frac{1}{2}E\left(\left|\sum_{r=1}^p\langle\Psi_{v_r}|[\tilde{L}_j,\tilde{L}_k]|\Psi_{v_r}\rangle\right|\right), \tag{17}$$

where $E(\cdot)$ denotes the expected value, each $|\Psi_{v_r}\rangle$ is randomly and independently chosen from the eigenvectors of ρ_x with a probability equal to the corresponding eigenvalue, i.e., each $|\Psi_{v_r}\rangle$ takes $|\Psi_q\rangle$ with probability $\lambda_q, q\in\{1,\ldots,m\}$. $\tilde{L}_j=\sum_q(F_Q^{-\frac{1}{2}})_{jq}L_q$ and $\tilde{L}_k=\sum_q(F_Q^{-\frac{1}{2}})_{kq}L_q$. For large p, this bound is almost as tight as the bound with $\frac{C_p}{p}$, the difference between $\frac{T_p}{p}$ and $\frac{C_p}{p}$ is at most of the order $O(\frac{1}{\sqrt{p}})$ with

$$\frac{(T_p)_{jk}}{p}\leqslant\frac{(C_p)_{jk}}{p}\leqslant\frac{(T_p)_{jk}}{p}+O\left(\frac{1}{\sqrt{p}}\right). \tag{18}$$

Asymptotically they converge to the same value,

$$\lim_{p\to\infty}\frac{(T_p)_{jk}}{p}=\lim_{p\to\infty}\frac{(C_p)_{jk}}{p}=\frac{1}{2}|\mathrm{Tr}(\rho_x[\tilde{L}_j,\tilde{L}_k])|. \tag{19}$$

(6) To demonstrate the versatility of the framework, we provide another set of bounds with the right logarithm derivative (RLD) operators.

$$\Gamma_p\leqslant\mathrm{Tr}\left[F_Q^{-1}F_{\mathrm{Re}}^{\mathrm{RLD}}\right]-\frac{1}{4(n-1)}\left\|\frac{C_p^{\mathrm{RLD}}}{p}\right\|_F^2, \tag{20}$$

where $F_{\mathrm{Re}}^{\mathrm{RLD}}$ is the real part of the RLD quantum Fisher information matrix with the jkth entry given by $(F^{\mathrm{RLD}})_{jk}=\mathrm{Tr}(\rho_xL_j^RL_k^{R\dagger})$, where L_j^R is the RLD operator corresponding to the parameter x_j, which can be obtained from $\partial_{x_j}\rho_x=\rho_xL_j^R$, $(C_p^{\mathrm{RLD}})_{jk}=\min\{\frac{1}{2}\|\sqrt{\rho_x^{\otimes p}}(\tilde{L}_{jp}^R\tilde{L}_{kp}^{R\dagger}-\tilde{L}_{kp}^R\tilde{L}_{jp}^{R\dagger})\sqrt{\rho_x^{\otimes p}}\|_1,2p\}$ with $\tilde{L}_{jp}^R=\sum_q(F_Q^{-\frac{1}{2}})_{jq}L_{qp}^R$ and $\tilde{L}_{kp}^R=\sum_q(F_Q^{-\frac{1}{2}})_{kq}L_{qp}^R$ with L_{qp}^R as the RLD operator of $\rho_x^{\otimes p}$ corresponding to the parameter x_q.

These bounds are in general not saturable, however, they are nontrivial regardless of the number of the parameters and the dimension of the quantum states. The upper bounds can also be directly transformed to the lower bounds for various other measures via the Cauchy-Schwarz inequality. For example, from the upper bound

$$\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\right]\leqslant n-\frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2, \tag{21}$$

we can obtain a lower bound for $\nu\,\mathrm{Tr}[F_Q\mathrm{Cov}(\hat{x})]$ via the Cauchy-Schwarz inequality as

$$\begin{aligned}\nu\,\mathrm{Tr}[F_Q\mathrm{Cov}(\hat{x})]&\geqslant\frac{n^2}{\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\right]}\\&\geqslant\frac{n^2}{n-\frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2}\\&\geqslant n+\frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2,\end{aligned} \tag{22}$$

which provides a lower bound on $\nu\,\mathrm{Tr}[F_Q\mathrm{Cov}(\hat{x})]$ under p-local measurements. $\nu\,\mathrm{Tr}[F_Q\mathrm{Cov}(\hat{x})]$ achieves the minimal value n when the QCRB is saturable and the gap between $\nu\,\mathrm{Tr}[F_Q\mathrm{Cov}(\hat{x})]$ and n provides a measure on the incompatibility. We note that the transformation from the upper bound to the lower bound via the Cauchy-Schwarz inequality does not work the other way, i.e., the lower bound on $\nu\,\mathrm{Tr}[F_Q\mathrm{Cov}(\hat{x})]$ can not be directly transformed to the upper bound on $\frac{1}{\nu}\mathrm{Tr}[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})]$ via the Cauchy-Schwarz inequality. This is one advantage of choosing $\frac{1}{\nu}\mathrm{Tr}[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})]$ over $\nu\,\mathrm{Tr}[F_Q\mathrm{Cov}(\hat{x})]$ as the measure of the incompatibility.

Similarly, we can obtain lower bounds on the weighted covariance matrix $\nu\,\mathrm{Tr}[W\,\mathrm{Cov}(\hat{x})]$, via the Cauchy-Schwarz inequality as

$$\nu\,\mathrm{Tr}[W\,\mathrm{Cov}(\hat{x})]\geqslant\frac{\left(\mathrm{Tr}\sqrt{F_Q^{-\frac{1}{2}}WF_Q^{-\frac{1}{2}}}\right)^2}{\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\right]}. \tag{23}$$

For example, from the upper bound

$$\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\right]\leqslant n-\frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2, \tag{24}$$

we can obtain a lower bound

$$\nu\,\mathrm{Tr}[W\,\mathrm{Cov}(\hat{x})]\geqslant\frac{\left(\mathrm{Tr}\sqrt{F_Q^{-\frac{1}{2}}WF_Q^{-\frac{1}{2}}}\right)^2}{n-\frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2}, \tag{25}$$

which constrains the precision that can be achieved under p-local measurements.

Aside from these analytical bounds, multiparameter precision bounds for mixed states, which require numerical optimization, are presented in Sec. IV A.

III. ANALYTICAL BOUNDS FOR PURE STATES

We start the derivation of the bounds for pure states, then generalize it to mixed states in the next section. Given a probe state $|\Psi_x\rangle$ with $x=(x_1,x_2,\ldots,x_n)$, and q operators $\{Y_1,Y_2,\ldots,Y_q\}$, we have

$$\begin{aligned}S&=(Y_1|\Psi_x\rangle\ \ldots\ Y_q|\Psi_x\rangle)^\dagger(Y_1|\Psi_x\rangle\ \ldots\ Y_q|\Psi_x\rangle)\\&\geqslant 0,\end{aligned} \tag{26}$$

where S is a $q\times q$ matrix with its jkth entry given by $(S)_{jk}=\langle\Psi_x|Y_j^\dagger Y_k|\Psi_x\rangle=\mathrm{Tr}(\rho_xY_j^\dagger Y_k)$ with $\rho_x=|\Psi_x\rangle\langle\Psi_x|$. We note that $S\geqslant 0$ also forms the basis for the generalized Robertson uncertainty relation [57,58].

We first consider a single copy of the state, for ν copies of the states, we can just replace $|\Psi(x)\rangle$ with $|\Psi(x)\rangle^{\otimes\nu}$. Given a measurement $\{M_\alpha\}$ with $\sum_\alpha M_\alpha = I$, we can construct n observables as

$$X_j = \sum_\alpha [\hat{x}_j(\alpha) - x_j] M_\alpha, \tag{27}$$

where $\hat{x}_j$ is the estimator for x_j. For locally unbiased estimator, we have

$$\mathrm{Tr}(\rho_x X_j) = 0, \quad j = 1, \ldots, n \tag{28}$$

and

$$\mathrm{Tr}(\partial_{x_j}\rho_x X_k) = \delta^j_k. \tag{29}$$

Let L_j be the SLD for x_j with $j \in \{1, \ldots, n\}$, then by replacing the set of $\{Y_j\}$ in Eq. (26) with the $2n$ operators, $\{X_1, \ldots, X_n, L_1, \ldots, L_n\}$, we have

$$S = \begin{pmatrix} A & B \\ B^\dagger & F \end{pmatrix} \geqslant 0, \tag{30}$$

where A, B, F are $n \times n$ matrices with the entries given by

$$\begin{aligned} (A)_{kj} &= \mathrm{Tr}(\rho_x X_k X_j), \\ (B)_{kj} &= \mathrm{Tr}(\rho_x X_k L_j), \\ (F)_{kj} &= \mathrm{Tr}(\rho_x L_k L_j). \end{aligned} \tag{31}$$

We can write these matrices in terms of the real and imaginary parts as $A = A_{\mathrm{Re}} + iA_{\mathrm{Im}}$, $B = B_{\mathrm{Re}} + iB_{\mathrm{Im}}$, $F = F_Q + iF_{\mathrm{Im}}$, where

$$\begin{aligned} (A_{\mathrm{Re}})_{kj} &= \tfrac{1}{2}\mathrm{Tr}(\rho_x\{X_k, X_j\}), \\ (B_{\mathrm{Re}})_{kj} &= \tfrac{1}{2}\mathrm{Tr}(\rho_x\{X_k, L_j\}), \\ (F_Q)_{kj} &= \tfrac{1}{2}\mathrm{Tr}(\rho_x\{L_k, L_j\}), \end{aligned} \tag{32}$$

where $\{X, Y\} = XY + YX$ is the anticommutator, and

$$\begin{aligned} (A_{\mathrm{Im}})_{kj} &= \frac{1}{2i}\mathrm{Tr}(\rho_x[X_k, X_j]), \\ (B_{\mathrm{Im}})_{kj} &= \frac{1}{2i}\mathrm{Tr}(\rho_x[X_k, L_j]), \\ (F_{\mathrm{Im}})_{kj} &= \frac{1}{2i}\mathrm{Tr}(\rho_x[L_k, L_j]), \end{aligned} \tag{33}$$

where $[X, Y] = XY - YX$ is the commutator. It is easy to see that F_Q is exactly the quantum Fisher information matrix, and the local unbiased condition in Eq. (29) can be equivalently written as

$$\mathrm{Tr}\big(\rho_x \tfrac{1}{2}\{L_j, X_k\}\big) = \delta^j_k, \tag{34}$$

which means $B_{\mathrm{Re}} = I$. A is the same as $Z(X)$ in the Holevo bound; however, we use a different notation here as in the case of mixed states it can be different from $Z(X)$.

As $\mathrm{Cov}(\hat{x}) \geqslant A$ [2,18,59], we have

$$\begin{pmatrix} \mathrm{Cov}(\hat{x}) & B \\ B^\dagger & F \end{pmatrix} = \begin{pmatrix} \mathrm{Cov}(\hat{x}) - A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A & B \\ B^\dagger & F \end{pmatrix} \geqslant 0. \tag{35}$$

Using the Schur's complement [60] we have

$$F - B^\dagger \mathrm{Cov}^{-1}(\hat{x}) B \geqslant 0, \tag{36}$$

this can be equivalently written as

$$\begin{aligned} &F_Q - \mathrm{Cov}^{-1}(\hat{x}) - B_{\mathrm{Im}}^T \mathrm{Cov}^{-1}(\hat{x}) B_{\mathrm{Im}} \\ &\quad + i\big[F_{\mathrm{Im}} + B_{\mathrm{Im}}^T \mathrm{Cov}^{-1}(\hat{x}) - \mathrm{Cov}^{-1}(\hat{x}) B_{\mathrm{Im}}\big] \geqslant 0. \end{aligned} \tag{37}$$

Since for a positive-semidefinite matrix $M \geqslant 0$, the real part is also positive semidefinite, i.e., $M_{\mathrm{Re}} = \frac{M+M^T}{2} \geqslant 0$. We thus have $F_Q - \mathrm{Cov}^{-1}(\hat{x}) - B_{\mathrm{Im}}^T \mathrm{Cov}^{-1}(\hat{x}) B_{\mathrm{Im}} \geqslant 0$, which can be equivalently written as

$$F_Q - \mathrm{Cov}^{-1}(\hat{x}) \geqslant B_{\mathrm{Im}}^T \mathrm{Cov}^{-1}(\hat{x}) B_{\mathrm{Im}}. \tag{38}$$

Note that $B_{\mathrm{Im}}^T \mathrm{Cov}^{-1}(\hat{x}) B_{\mathrm{Im}} \geqslant 0$, thus the real part of Eq. (37) already gives a tighter bound than the QCRB.

By multiplying $F_Q^{-\frac{1}{2}}$ from both the left and the right of Eq. (37), we get

$$\begin{aligned} &I - \tilde{\mathrm{Cov}}^{-1}(\hat{x}) - \tilde{B}_{\mathrm{Im}}^T \tilde{\mathrm{Cov}}^{-1}(\hat{x}) \tilde{B}_{\mathrm{Im}} \\ &\quad + i\big[\tilde{F}_{\mathrm{Im}} + \tilde{B}_{\mathrm{Im}}^T \tilde{\mathrm{Cov}}^{-1}(\hat{x}) - \tilde{\mathrm{Cov}}^{-1}(\hat{x}) \tilde{B}_{\mathrm{Im}}\big] \geqslant 0, \end{aligned} \tag{39}$$

here $\tilde{\mathrm{Cov}}^{-1}(\hat{x}) = F_Q^{-\frac{1}{2}} \mathrm{Cov}^{-1}(\hat{x}) F_Q^{-\frac{1}{2}}$, $\tilde{B}_{\mathrm{Im}} = F_Q^{\frac{1}{2}} B_{\mathrm{Im}} F_Q^{-\frac{1}{2}}$, $\tilde{F}_{\mathrm{Im}} = F_Q^{-\frac{1}{2}} F_{\mathrm{Im}} F_Q^{-\frac{1}{2}}$. This is equivalent to the reparametrization which changes the QFIM to the identity, and $\tilde{\mathrm{Cov}}(\hat{x})$ can be regarded as the covariance matrix under the reparametrization. Various tradeoff relations can be obtained from Eq. (39). In the Appendix A, we show that Eq. (39) implies

$$1 - [\tilde{\mathrm{Cov}}^{-1}(\hat{x})]_{jj} + 1 - [\tilde{\mathrm{Cov}}^{-1}(\hat{x})]_{kk} \geqslant \tfrac{1}{2}|(\tilde{F}_{\mathrm{Im}})_{jk}|^2. \tag{40}$$

This describes a tradeoff between $[\tilde{\mathrm{Cov}}^{-1}(\hat{x})]_{jj}$ and $[\tilde{\mathrm{Cov}}^{-1}(\hat{x})]_{kk}$ as they can not reach 1 simultaneously when $(\tilde{F}_{\mathrm{Im}})_{jk} \neq 0$.

By summing Eq. (40) over all different choices of j, k, we can obtain

$$\begin{aligned} \mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] &= \mathrm{Tr}[\tilde{\mathrm{Cov}}^{-1}(\hat{x})] \\ &\leqslant n - \frac{1}{4(n-1)} \big\| F_Q^{-\frac{1}{2}} F_{\mathrm{Im}} F_Q^{-\frac{1}{2}} \big\|_F^2, \end{aligned} \tag{41}$$

where $\|\cdot\|_F = \sqrt{\sum_{j,k} |(\cdot)_{jk}|^2}$ is the Frobenius norm.

When there are ν copies of the state, we can replace $|\Psi(x)\rangle$ with $|\Psi(x)\rangle^{\otimes\nu}$ and repeat the procedure to get the tradeoff relation as

$$\begin{aligned} &\mathrm{Tr}\big[F_{Q\nu}^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] \\ &\quad \leqslant n - \frac{1}{4(n-1)} \big\| F_{Q\nu}^{-\frac{1}{2}} F_{\mathrm{Im}\nu} F_{Q\nu}^{-\frac{1}{2}} \big\|_F^2, \end{aligned} \tag{42}$$

where $F_\nu = F_{Q\nu} + iF_{\mathrm{Im}\nu}$ is the corresponding operator associated with $|\Psi(x)\rangle^{\otimes\nu}$. It is easy to verify that $F_{Q\nu} = \nu F_Q$, which is the QFIM for $|\Psi(x)\rangle^{\otimes\nu}$, and $F_{\mathrm{Im}\nu} = \nu F_{\mathrm{Im}}$. Thus, when there are ν copies of the state, the tradeoff relation is given by

$$\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\right] \leqslant n-\frac{1}{4(n-1)}\left\|F_Q^{-\frac{1}{2}}F_{\mathrm{Im}}F_Q^{-\frac{1}{2}}\right\|_F^2, \quad (43)$$

This tradeoff relation holds for arbitrary measurements on ν copies of the states.

The tradeoff relation for ν copies of the pure state can also be obtained in an alternative way. Note that for pure states the optimal measurement can be taken as the 1-local measurement [15], if we repeat the 1-local measurement ν times with ν copies of the state, $\mathrm{Cov}(\hat{x})$ will be reduced by ν times. Equation (41), which is the tradeoff relation for a single state, then directly becomes Eq. (43) since $\mathrm{Cov}(\hat{x})$ is reduced by ν times. The two ways to get Eq. (43), however, have different meanings. The derivation that uses $|\Psi(x)\rangle^{\otimes\nu}$ allows arbitrary measurement on $|\Psi(x)\rangle^{\otimes\nu}$ while the derivation with the repetition of the 1-local measurement only uses 1-local measurement. The reason that they lead to the same tradeoff relation is that for pure states 1-local measurement is already optimal, allowing collective measurements does not improve the precision. The situation is different for mixed states as we will see in the next section.

The bound in Eq. (43) is obtained by summing the tradeoff relations between pairs of parameters in Eq. (40), which ignores the correlations with the other parameters. The presence of other parameters, however, can affect the precisions. In the Appendix A we show that when $n \geqslant 3$, by including the correlations among the parameters, the bound can be improved as

$$\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\right] \leqslant n-\frac{n-2}{(n-1)^2}\left\|F_Q^{-\frac{1}{2}}F_{\mathrm{Im}}F_Q^{-\frac{1}{2}}\right\|_F^2. \quad (44)$$

Since $\frac{n-2}{n-1} > \frac{1}{4}$ when $n \geqslant 3$, this is tighter than the bound in Eq. (43). It is also tighter than summing the tightest bound for a pair of parameters in previous study [31].

We can obtain even tighter tradeoff relation for large n as (see Appendix A for detailed derivation)

$$\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\right] \leqslant n-\frac{1}{5}\left\|F_Q^{-\frac{1}{2}}F_{\mathrm{Im}}F_Q^{-\frac{1}{2}}\right\|_F^2, \quad (45)$$

which is tighter than Eq. (44) when $n \geqslant 5$.

The three bounds in Eqs. (43)–(45) can be written concisely as

$$\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\right] \leqslant n-f(n)\left\|F_Q^{-\frac{1}{2}}F_{\mathrm{Im}}F_Q^{-\frac{1}{2}}\right\|_F^2, \quad (46)$$

where $f(n)=\frac{1}{4(n-1)}$, $\frac{n-2}{(n-1)^2}$ or $\frac{1}{5}$. These bounds are all valid for any n. Since larger $f(n)$ leads to tighter bound we can take $f(n)=\max\{\frac{1}{4(n-1)}, \frac{n-2}{(n-1)^2}, \frac{1}{5}\}$ to get a tighter upper bound.

IV. PRECISION BOUNDS FOR MIXED STATES

For pure states, the ultimate precision under the local measurement can be quantified by the Holevo bound since for pure states the Holevo bound can be saturated with the 1-local measurement. For mixed states, however, the Holevo bound is in general not saturable under the local measurement. We will first provide a tighter bound for the mixed states under local measurement, then use it to obtain the upper bounds for the incompatibility measures.

A. Multiparameter precision bound for mixed states

For a mixed state ρ_x, with $x=(x_1,\ldots,x_n)$, given any POVM, $\{M_\alpha\}$, and any $|u\rangle$, we define Cov_u as a $n\times n$ matrix with the jkth entry given by

$$(\mathrm{Cov}_u)_{jk}=\sum_\alpha[\hat{x}_j(\alpha)-x_j][\hat{x}_k(\alpha)-x_k]\langle u|\sqrt{\rho_x}M_\alpha\sqrt{\rho_x}|u\rangle, \quad (47)$$

and A_u as a $n\times n$ matrix with the jkth entry given by

$$\begin{aligned}(A_u)_{jk} &= \langle u|\sqrt{\rho_x}X_j^\dagger X_k\sqrt{\rho_x}|u\rangle \\ &= \frac{1}{2}\langle u|\sqrt{\rho_x}\{X_j,X_k\}\sqrt{\rho_x}|u\rangle \\ &\quad + i\frac{1}{2i}\langle u|\sqrt{\rho_x}[X_j,X_k]\sqrt{\rho_x}|u\rangle, \end{aligned} \quad (48)$$

where $X_j=\sum_\alpha[\hat{x}_j(\alpha)-x_j]M_\alpha$ is locally unbiased.

We first note that for any set of $\{|u_q\rangle\}$ that satisfies $\sum_q|u_q\rangle\langle u_q|=I$, we have $\mathrm{Cov}(\hat{x})=\sum_q\mathrm{Cov}_{u_q}$. This can be verified by comparing $\sum_q(\mathrm{Cov}_{u_q})_{jk}$ and $\mathrm{Cov}(\hat{x})_{jk}$ as

$$\begin{aligned}&\sum_q\left(\mathrm{Cov}_{u_q}\right)_{jk} \\ &= \sum_q\sum_\alpha[\hat{x}_j(\alpha)-x_j][\hat{x}_k(\alpha)-x_k]\langle u_q|\sqrt{\rho_x}M_\alpha\sqrt{\rho_x}|u_q\rangle \\ &= \sum_\alpha[\hat{x}_j(\alpha)-x_j][\hat{x}_k(\alpha)-x_k]\mathrm{Tr}(\rho_xM_\alpha) \\ &= \mathrm{Cov}(\hat{x})_{jk}. \end{aligned} \quad (49)$$

And for any $|u\rangle$, we have $\mathrm{Cov}_u \geqslant A_u$(see Appendix D).

Since Cov_u is symmetric, we also have $\mathrm{Cov}_u=\mathrm{Cov}_u^T \geqslant A_u^T$. Thus, for any set of $\{|u_q\rangle\}$ that satisfies $\sum_q|u_q\rangle\langle u_q|=I$ and any choices of $\bar{\mathbf{A}}_{u_q}\in\{A_{u_q},A_{u_q}^T\}$, we have

$$\mathrm{Cov}(\hat{x})=\sum_q\mathrm{Cov}_{u_q}\geqslant\bar{\mathbf{A}}=\sum_q\bar{\mathbf{A}}_{u_q}, \quad (50)$$

where $\bar{\mathbf{A}}_{u_q}$ equal to either A_{u_q} or $A_{u_q}^T$. We can write $\bar{\mathbf{A}}$ in terms of the real and imaginary parts as $\bar{\mathbf{A}}=\bar{\mathbf{A}}_{\mathrm{Re}}+i\bar{\mathbf{A}}_{\mathrm{Im}}$, then

$$\nu\,\mathrm{Tr}[W\,\mathrm{Cov}(\hat{x})]\geqslant\min_{\{X_j\}}\mathrm{Tr}[W\bar{\mathbf{A}}_{\mathrm{Re}}]+\|\sqrt{W}\bar{\mathbf{A}}_{\mathrm{Im}}\sqrt{W}\|_1, \quad (51)$$

where $W \geqslant 0$ is the weight matrix and the number of repetition ν has been included.

This includes the Holevo bound [2] and the Nagaoka bound [51,52] as special cases. To see the connection with the Holevo bound, we just choose $\bar{\mathbf{A}}_{u_q}=A_{u_q}$ for all q, then for any set of $\{|u_q\rangle\}$ that satisfies $\sum_q|u_q\rangle\langle u_q|=I$, we have $\bar{\mathbf{A}}=\sum_qA_{u_q}=Z(X)$ since (note that X_j is Hermitian)

$$\begin{aligned}\bar{\mathbf{A}}_{jk} &= \sum_q\left(A_{u_q}\right)_{jk} \\ &= \sum_q\langle u_q|\sqrt{\rho_x}X_j^\dagger X_k\sqrt{\rho_x}|u_q\rangle \\ &= \mathrm{Tr}(\rho_xX_j^\dagger X_k) \\ &= Z(X)_{jk}. \end{aligned} \quad (52)$$

Equation (51) then reduces to the Holevo bound. When there are only two parameters x_1 and x_2, we can choose the set of $\{|u_q\rangle\}$ as the eigenvectors of $\sqrt{\rho_x}[X_1, X_2]\sqrt{\rho_x}$ and choose $\bar{\mathbf{A}}_{u_q}$ as

$$\bar{\mathbf{A}}_{u_q} := \begin{cases} A_{u_q} & \text{for } \frac{1}{2i}\langle u_q|\sqrt{\rho_x}[X_1, X_2]\sqrt{\rho_x}|u_q\rangle \geqslant 0, \\ A_{u_q}^T & \text{for } \frac{1}{2i}\langle u_q|\sqrt{\rho_x}[X_1, X_2]\sqrt{\rho_x}|u_q\rangle < 0. \end{cases} \quad (53)$$

Intuitively, A_{u_q} can be written as the real and imaginary parts as $A_{u_q} = A_{u_q\mathrm{Re}} + iA_{u_q\mathrm{Im}}$, where $A_{u_q\mathrm{Im}}$ is a 2×2 skew symmetric matrix $\begin{pmatrix} 0 & a_q \\ -a_q & 0 \end{pmatrix}$ with $a_q = \frac{1}{2i}\langle u_q|\sqrt{\rho_x}[X_1, X_2]\sqrt{\rho_x}|u_q\rangle$. $\bar{\mathbf{A}}_{u_q}$ is then chosen according to the sign of a_q, $\bar{\mathbf{A}}_{u_q} = A_{u_q}$ when $a_q \geqslant 0$ and $\bar{\mathbf{A}}_{u_q} = A_{u_q}^T$ when $a_q \leqslant 0$. The imaginary parts of different $\bar{\mathbf{A}}_{u_q}$ are then aligned and add up to $\frac{1}{2}\|\sqrt{\rho_x}[X_1, X_2]\sqrt{\rho_x}\|_1$. With this choice we then have

$$\begin{aligned} \bar{\mathbf{A}} &= \sum_q \bar{\mathbf{A}}_{u_q} \\ &= \begin{pmatrix} \mathrm{Tr}(\rho_x X_1^2) & \frac{1}{2}\mathrm{Tr}[\rho_x\{X_1, X_2\}] \\ \frac{1}{2}\mathrm{Tr}[\rho_x\{X_1, X_2\}] & \mathrm{Tr}(\rho_x X_2^2) \end{pmatrix} \\ &\quad + i\begin{pmatrix} 0 & \frac{1}{2}\|\sqrt{\rho_x}[X_1, X_2]\sqrt{\rho_x}\|_1 \\ -\frac{1}{2}\|\sqrt{\rho_x}[X_1, X_2]\sqrt{\rho_x}\|_1 & 0 \end{pmatrix}. \end{aligned} \quad (54)$$

Equation (51) then becomes (with $W = I$)

$$\begin{aligned} &\nu\, \mathrm{Tr}[\mathrm{Cov}(\hat{x})] \\ &\quad \geqslant \min_{\{X_1, X_2\}} \mathrm{Tr}[\bar{\mathbf{A}}_{\mathrm{Re}}] + \|\bar{\mathbf{A}}_{\mathrm{Im}}\|_1 \\ &\quad = \min_{\{X_1, X_2\}} \mathrm{Tr}(\rho_x X_1^2) + \mathrm{Tr}(\rho_x X_2^2) + \|\sqrt{\rho_x}[X_1, X_2]\sqrt{\rho_x}\|_1, \end{aligned} \quad (55)$$

which recovers the Nagaoka bound [51,52]. This establishes a connection between the Holevo bound and the Nagaoka bound and improves our understanding on these existing bounds.

The optimal choice of $|u_q\rangle$ and $\bar{\mathbf{A}}_{u_q}$ provides the tightest bound, but any choice leads to a valid bound. We now show how nontrivial analytical upper bounds on Γ_p can be obtained by making particular choices of $|u_q\rangle$ and $\bar{\mathbf{A}}_{u_q}$.

B. Incompatibility under 1-local measurements

Given a mixed state ρ_x, we can make a reparametrization with $\tilde{x} = F_Q^{\frac{1}{2}}x$ under which $\tilde{F}_Q = I$, and $\tilde{\mathrm{Cov}}(\hat{x}) = F_Q^{\frac{1}{2}}\mathrm{Cov}(\hat{x})F_Q^{\frac{1}{2}}$. Thus, without loss of generality, we start with the case that the QFIM equals to the identity.

We first consider the precision under 1-local measurements, i.e., separable measurements. Note that for any vector $|u\rangle$, we have

$$\begin{aligned} S_u &= (X_1\sqrt{\rho_x}|u\rangle \ \ldots\ X_n\sqrt{\rho_x}|u\rangle\ L_1\sqrt{\rho_x}|u\rangle\ \ldots\ L_n\sqrt{\rho_x}|u\rangle)^\dagger \\ &\quad (X_1\sqrt{\rho_x}|u\rangle \ \ldots\ X_n\sqrt{\rho_x}|u\rangle\ L_1\sqrt{\rho_x}|u\rangle\ \ldots\ L_n\sqrt{\rho_x}|u\rangle) \\ &= \begin{pmatrix} A_u & B_u \\ B_u^\dagger & F_u \end{pmatrix} \geqslant 0, \end{aligned} \quad (56)$$

where A_u, B_u, and F_u are $n \times n$ matrices with the entries given by

$$\begin{aligned} (A_u)_{jk} &= \langle u|\sqrt{\rho_x}X_j^\dagger X_k\sqrt{\rho_x}|u\rangle \\ &= \frac{1}{2}\langle u|\sqrt{\rho_x}\{X_j, X_k\}\sqrt{\rho_x}|u\rangle \\ &\quad + i\frac{1}{2i}\langle u|\sqrt{\rho_x}[X_j, X_k]\sqrt{\rho_x}|u\rangle, \\ (B_u)_{jk} &= \langle u|\sqrt{\rho_x}X_j^\dagger L_k\sqrt{\rho_x}|u\rangle \\ &= \frac{1}{2}\langle u|\sqrt{\rho_x}\{X_j, L_k\}\sqrt{\rho_x}|u\rangle \\ &\quad + i\frac{1}{2i}\langle u|\sqrt{\rho_x}[X_j, L_k]\sqrt{\rho_x}|u\rangle, \\ (F_u)_{jk} &= \langle u|\sqrt{\rho_x}L_j^\dagger L_k\sqrt{\rho_x}|u\rangle \\ &= \frac{1}{2}\langle u|\sqrt{\rho_x}\{L_j, L_k\}\sqrt{\rho_x}|u\rangle \\ &\quad + i\frac{1}{2i}\langle u|\sqrt{\rho_x}[L_j, L_k]\sqrt{\rho_x}|u\rangle. \end{aligned} \quad (57)$$

For a set of $\{|u_q\rangle\}$ with $\sum_q |u_q\rangle\langle u_q| = I$, we obtain a corresponding set of $\{S_{u_q}\}$. We then let $\bar{\mathbf{S}} = \sum_q \bar{\mathbf{S}}_{u_q}$ where $\bar{\mathbf{S}}_{u_q} \in \{S_{u_q}, S_{u_q}^T\}$. Since $S_{u_q} \geqslant 0$ and $S_{u_q}^T \geqslant 0$, it is then easy to see that

$$\bar{\mathbf{S}} = \sum_q \bar{\mathbf{S}}_{u_q} = \begin{pmatrix} \bar{\mathbf{A}} & \bar{\mathbf{B}} \\ \bar{\mathbf{B}}^\dagger & \bar{\mathbf{F}} \end{pmatrix} \geqslant 0, \quad (58)$$

where $\bar{\mathbf{F}} = \sum_q \bar{\mathbf{F}}_{u_q}$ with $\bar{\mathbf{F}}_{u_q}$ equal to either F_{u_q} or $F_{u_q}^T$, $\bar{\mathbf{A}} = \sum_q \bar{\mathbf{A}}_{u_q}$ with $\bar{\mathbf{A}}_{u_q}$ equals to either A_{u_q} or $A_{u_q}^T$, and $\bar{\mathbf{B}} = \sum_q \bar{\mathbf{B}}_{u_q}$. Since $\bar{\mathbf{S}}_{u_q}$ has the same real part as S_{u_q}, the real part of $\bar{\mathbf{S}}$ is independent of the choices of $\bar{\mathbf{S}}_{u_q}$. In particular, the real part of $\bar{\mathbf{F}}$ always equals to the QFIM as

$$\begin{aligned} (\bar{\mathbf{F}}_{\mathrm{Re}})_{jk} &= \sum_q \frac{1}{2}\langle u_q|\sqrt{\rho_x}\{L_j, L_k\}\sqrt{\rho_x}|u_q\rangle \\ &= \mathrm{Tr}\left[\rho_x\frac{1}{2}\{L_j, L_k\}\right] \\ &= (F_Q)_{jk}. \end{aligned} \quad (59)$$

Similarly, it is straightforward to see that the real part of $\bar{\mathbf{B}}$ also remains the same as

$$\begin{aligned} (\bar{\mathbf{B}}_{\mathrm{Re}})_{jk} &= \sum_q \frac{1}{2}\langle u_q|\sqrt{\rho_x}\{X_j, L_k\}\sqrt{\rho_x}|u_q\rangle \\ &= \mathrm{Tr}\left[\rho_x\frac{1}{2}\{X_j, L_k\}\right] \\ &= \delta_k^j, \end{aligned} \quad (60)$$

where the last equality is the locally unbiased condition. We can thus write $\bar{\mathbf{B}} = I + i\bar{\mathbf{B}}_{\mathrm{Im}}$. Since $\mathrm{Cov}(\hat{x}) \geqslant \bar{\mathbf{A}}$, we have

$$\begin{pmatrix} \mathrm{Cov}(\hat{x}) & \bar{\mathbf{B}} \\ \bar{\mathbf{B}}^\dagger & \bar{\mathbf{F}} \end{pmatrix} \geqslant 0. \quad (61)$$

Then by following the same derivation as in the previous section we have

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] \leqslant n - f(n)\big\|F_Q^{-\frac{1}{2}}\bar{\mathbf{F}}_{\mathrm{Im}}F_Q^{-\frac{1}{2}}\big\|_F^2, \tag{62}$$

where $f(n)=\max\{\frac{1}{4(n-1)},\frac{n-2}{(n-1)^2},\frac{1}{5}\}$, $\bar{\mathbf{F}}_{\mathrm{Im}}$ is the imaginary part of $\bar{\mathbf{F}}=\sum_q \bar{\mathbf{F}}_{u_q}$ with each $\bar{\mathbf{F}}_{u_q}$ equals to either F_{u_q} or $F_{u_q}^T$ which can be optimized to get the maximal $\|F_Q^{-\frac{1}{2}}\bar{\mathbf{F}}_{\mathrm{Im}}F_Q^{-\frac{1}{2}}\|_F$.

We can also obtain additional bounds by combining different choices of $\{|u_q\rangle\}$. In particular, we can choose different set of $\{|u_q\rangle\}$ according to different pair of indices, say $\alpha\neq\beta\in\{1,2,\dots,n\}$. Specifically, for a given pair of indices, α and β, we choose a set of $\{|u_1\rangle,\dots,|u_d\rangle\}$ as the orthonormal eigenvectors of $\sqrt{\rho_x}[L_\alpha,L_\beta]\sqrt{\rho_x}$. Note that $\sqrt{\rho_x}[L_\alpha,L_\beta]\sqrt{\rho_x}$ is skew Hermitian whose eigenvalues are pure imaginary, thus for any eigenvector $|u_q\rangle$, $\langle u_q|\sqrt{\rho_x}[L_\alpha,L_\beta]\sqrt{\rho_x}|u_q\rangle = ia_q$ with a_q a real number. The imaginary axis of $(F_{u_q})_{\alpha\beta} = \langle u_q|\sqrt{\rho_x}L_\alpha L_\beta\sqrt{\rho_x}|u_q\rangle$ is then $\frac{1}{2i}\langle u_q|\sqrt{\rho_x}[L_\alpha,L_\beta]\sqrt{\rho_x}|u_q\rangle = \frac{1}{2}a_q$. We then let

$$\bar{\mathbf{S}}_{u_q} := \begin{cases} S_{u_q} & \text{for } a_q\geqslant 0,\\ S_{u_q}^T & \text{for } a_q<0,\end{cases} \tag{63}$$

and sum $\bar{\mathbf{S}}_{u_q}$ to get

$$\bar{\mathbf{S}} = \sum_q \bar{\mathbf{S}}_{u_q} = \begin{pmatrix}\bar{\mathbf{A}} & \bar{\mathbf{B}}\\ \bar{\mathbf{B}}^\dagger & \bar{\mathbf{F}}\end{pmatrix}\geqslant 0, \tag{64}$$

where $\bar{\mathbf{F}}=\sum_q\bar{\mathbf{F}}_{u_q}$ with $\bar{\mathbf{F}}_{u_q}$ equal to either F_{u_q} or $F_{u_q}^T$ which are determined by the choices in Eq. (63) so that the imaginary parts of all $(\bar{\mathbf{F}}_{u_q})_{\alpha\beta}$ are all positive, $\bar{\mathbf{A}}=\sum_q\bar{\mathbf{A}}_{u_q}$ with $\bar{\mathbf{A}}_{u_q}$ equals to either A_{u_q} or $A_{u_q}^T$, and $\bar{\mathbf{B}}=\sum_q\bar{\mathbf{B}}_{u_q}$. It is easy to verify that according to the choices in Eq. (63), which aligns the imaginary part of the $\alpha\beta$th entry of each $\bar{\mathbf{F}}_{u_q}$ with the same sign, we have

$$(\bar{\mathbf{F}}_{\mathrm{Im}})_{\alpha\beta} = \sum_q\frac{1}{2}|a_q| = \frac{1}{2}\|\sqrt{\rho_x}[L_\alpha,L_\beta]\sqrt{\rho_x}\|_1, \tag{65}$$

where $\|\dots\|_1$ is the trace norm which equals to the sum of singular values and for the skew Hermitian matrix just equals to the sum of the absolute value of the eigenvalues. Again, since $\mathrm{Cov}(\hat{x})\geqslant\bar{\mathbf{A}}$, we have

$$\begin{pmatrix}\mathrm{Cov}(\hat{x}) & \bar{\mathbf{B}}\\ \bar{\mathbf{B}}^\dagger & \bar{\mathbf{F}}\end{pmatrix}\geqslant 0. \tag{66}$$

Then by following the same derivation as in the previous section, under the parametrization that $F_Q=I$, we can get the same tradeoff relation, similar as Eq. (40). Specifically, for the entries associated with α and β, we have

$$1-[\mathrm{Cov}^{-1}(\hat{x})]_{\alpha\alpha}+1-[\mathrm{Cov}^{-1}(\hat{x})]_{\beta\beta}\geqslant \frac{1}{2}|(\bar{\mathbf{F}}_{\mathrm{Im}})_{\alpha\beta}|^2 \tag{67}$$

with $(\bar{\mathbf{F}}_{\mathrm{Im}})_{\alpha\beta}=\frac{1}{2}\|\sqrt{\rho_x}[L_\alpha,L_\beta]\sqrt{\rho_x}\|_1$. We note that here we make the choices of $\{|u_q\rangle\}$ and $\{\bar{\mathbf{S}}_{u_q}\}$ according to a particular pair of indices α and β, thus only the imaginary part of $(\bar{\mathbf{F}}_{\mathrm{Im}})_{\alpha\beta}$ equals to $\frac{1}{2}\|\sqrt{\rho_x}[L_\alpha,L_\beta]\sqrt{\rho_x}\|_1$, for other indices $(j,k)\neq(\alpha,\beta)$, in general $(\bar{\mathbf{F}}_{\mathrm{Im}})_{jk}\neq\frac{1}{2}\|\sqrt{\rho_x}[L_j,L_k]\sqrt{\rho_x}\|_1$. However, for different pairs of indices, we can repeat the procedure, i.e., choose another set of $\{|u_q\rangle\}$ and $\bar{\mathbf{S}}_{u_q}$, to get the same tradeoff relations with different indices as

$$1-[\mathrm{Cov}^{-1}(\hat{x})]_{jj}+1-[\mathrm{Cov}^{-1}(\hat{x})]_{kk} \geqslant \frac{1}{2}\big(\tfrac{1}{2}\|\sqrt{\rho_x}[L_j,L_k]\sqrt{\rho_x}\|_1\big)^2. \tag{68}$$

We note that these tradeoff relations are on the same covariance matrix as the choices of $\{|u_q\rangle\}$ and $\bar{\mathbf{S}}_{u_q}$ do not affect the covariance matrix itself, they are only used to obtain the bounds.

By summing the tradeoff relations in Eq. (68) over all pairs of indices we get

$$\frac{1}{\nu}\mathrm{Tr}[\mathrm{Cov}^{-1}(\hat{x})]\leqslant n-\frac{1}{4(n-1)}\|C_1\|_F^2, \tag{69}$$

where ν comes from repeating the 1-local measurement on ν copies of the state, C_1 is a matrix with its entries given by

$$(C_1)_{jk}=\tfrac{1}{2}\|\sqrt{\rho_x}[L_j,L_k]\sqrt{\rho_x}\|_1. \tag{70}$$

We note that C_1 is different from any particular $\bar{\mathbf{F}}_{\mathrm{Im}}$. We get different $\bar{\mathbf{F}}_{\mathrm{Im}}$ by choosing different $\{|u_q\rangle\}$ and $\bar{\mathbf{S}}_{u_q}$ for different pairs of indices. C_1 is obtained by combining the tradeoff relations in Eq. (68) which are obtained by choosing different $\{|u_q\rangle\}$ and $\bar{\mathbf{S}}_{u_q}$ for different indices.

As stated at the beginning of this section, when $F_Q\neq I$ in the original parametrization, we can make a reparametrization $\tilde{x}=F_Q^{\frac{1}{2}}x$, under which $\tilde{F}_Q=I$, $\tilde{\mathrm{Cov}}(\hat{\tilde{x}})=F_Q^{\frac{1}{2}}\mathrm{Cov}(\hat{x})F_Q^{\frac{1}{2}}$, $\tilde{L}_j=\sum_q(F_Q^{-\frac{1}{2}})_{jq}L_q$, the tradeoff relation in Eq. (69) can be written in the original parametrization as

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big]=\frac{1}{\nu}\mathrm{Tr}[\tilde{\mathrm{Cov}}^{-1}(\hat{x})] \leqslant n-\frac{1}{4(n-1)}\|C_1\|_F^2, \tag{71}$$

with the entries of C_1 given by

$$(C_1)_{jk}=\frac{1}{2}\|\sqrt{\rho_x}[\tilde{L}_j,\tilde{L}_k]\sqrt{\rho_x}\|_1 = \frac{1}{2}\left\|\sqrt{\rho_x}\left[\sum_q\big(F_Q^{-\frac{1}{2}}\big)_{jq}L_q,\sum_q\big(F_Q^{-\frac{1}{2}}\big)_{kq}L_q\right]\sqrt{\rho_x}\right\|_1. \tag{72}$$

The tradeoff relation immediately gives a necessary condition for the saturation of the QCRB under the 1-local measurement. To saturate the QCRB, i.e., for $\mathrm{Cov}(\hat{x})=\frac{1}{\nu}F_Q^{-1}$, it requires $\frac{1}{\nu}\mathrm{Tr}[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})]=n$, which is only possible when $C_1=0$, i.e., $\sqrt{\rho_x}[\tilde{L}_j,\tilde{L}_k]\sqrt{\rho_x}=0$ for any j,k. This is the partial commutative condition expressed under the parametrization where $\tilde{F}_Q=I$, and is equivalent to the partial commutative condition in the original parametrization as $\sqrt{\rho_x}[L_q,L_s]\sqrt{\rho_x}=0$ for any q and s. The equivalence can be seen by writing $L_q=\sum_j(F_Q^{\frac{1}{2}})_{qj}\tilde{L}_j$ and $L_s=\sum_k(F_Q^{\frac{1}{2}})_{sk}\tilde{L}_k$; it is then easy to see that when $\sqrt{\rho_x}[\tilde{L}_j,\tilde{L}_k]\sqrt{\rho_x}=0$ for any j,k we have $\sqrt{\rho_x}[L_q,L_s]\sqrt{\rho_x}=0$ for any q and s, and vice versa.

C. Incompatibility measures under p-local measurements

For p-local measurements, which are the collective measurements on at most p copies of the state, we can get the tradeoff relation by replacing ρ_x with $\rho_x^{\otimes p}$ in the previous section. Again we first assume $F_Q = I$ for ρ_x, then $F_{Qp} = pI$ for $\rho_x^{\otimes p}$. Following the same procedure as the previous section, for a fixed pair of j, k, by substituting $\tilde{\text{Cov}}^{-1}(\hat{x}) = F_{Qp}^{-\frac{1}{2}}\text{Cov}^{-1}(\hat{x})F_{Qp}^{-\frac{1}{2}} = \frac{\text{Cov}^{-1}(\hat{x})}{p}$ and $\tilde{F}_{\text{Im}p} = F_{Qp}^{-\frac{1}{2}}\bar{\mathbf{F}}_{\text{Im}p}F_{Qp}^{-\frac{1}{2}} = \frac{\bar{\mathbf{F}}_{\text{Im}p}}{p}$ in Eq. (40) we can get

$$1 - \frac{\text{Cov}^{-1}(\hat{x})_{jj}}{p} + 1 - \frac{\text{Cov}^{-1}(\hat{x})_{kk}}{p} \geqslant \frac{1}{2}\left|\frac{(\bar{\mathbf{F}}_{\text{Im}p})_{jk}}{p}\right|^2 \quad (73)$$

with $(\bar{\mathbf{F}}_{\text{Im}p})_{jk} = \frac{1}{2}\|\sqrt{\rho_x^{\otimes p}}[L_{jp}, L_{kp}]\sqrt{\rho_x^{\otimes p}}\|_1$, where L_{jp} is the SLD corresponding to the parameter x_j for $\rho_x^{\otimes p}$, which can be written as $L_{jp} = \sum_{r=1}^{p} L_j^{(r)}$ with $L_j^{(r)} = I^{\otimes(r-1)} \otimes L_j \otimes I^{\otimes(p-r)}$, $r = 1, \dots, p$, L_j is the SLD for a single copy of the state.

Again, we can repeat the procedure for different pairs of j, k and sum over all pairs of j, k to get the tradeoff relation. Under the parametrization that $F_Q = I$, we have

$$\frac{1}{\nu/p}\frac{\text{Tr}[\text{Cov}^{-1}(\hat{x})]}{p} = \frac{1}{\nu}\text{Tr}[\text{Cov}^{-1}(\hat{x})] \leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2, \quad (74)$$

where the factor $\frac{1}{\nu/p}$ comes from repeating the p-local measurement ν/p times on a total ν copies of the state, C_p is a matrix with the entries given by

$$(C_p)_{jk} = \frac{1}{2}\left\|\sqrt{\rho_x^{\otimes p}}[L_{jp}, L_{kp}]\sqrt{\rho_x^{\otimes p}}\right\|_1. \quad (75)$$

If $F_Q \neq I$ in the initial parametrization, we can again make a reparametrization $\tilde{x} = F_Q^{\frac{1}{2}}x$ first, under which $\tilde{L}_j = \sum_q (F_Q^{-\frac{1}{2}})_{jq}L_q$. The tradeoff relation can then be written as

$$\frac{1}{\nu}\text{Tr}\big[F_Q^{-1}\text{Cov}^{-1}(\hat{x})\big] \leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2, \quad (76)$$

with

$$(C_p)_{jk} = \tfrac{1}{2}\left\|\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{jp}, \tilde{L}_{kp}]\sqrt{\rho_x^{\otimes p}}\right\|_1, \quad (77)$$

where $\tilde{L}_{jp} = \sum_q (F_Q^{-\frac{1}{2}})_{jq}L_{qp}$.

$\|\frac{C_p}{p}\|_F$ determines the gap between the bound and n, which measures the incompatibility of the measurements. Since p-local measurement is a subset of $(p+1)$-local measurement, we expect that $\|\frac{C_{p+1}}{p+1}\|_F \leqslant \|\frac{C_p}{p}\|_F$ since there should be less incompatibility when more measurements are allowed. This can be verified as

$$\begin{aligned}
&\frac{\left\|\sqrt{\rho_x^{\otimes(p+1)}}[\tilde{L}_{j(p+1)}, \tilde{L}_{k(p+1)}]\sqrt{\rho_x^{\otimes(p+1)}}\right\|_1}{p+1} \\
&= \frac{\left\|\sum_{r=1}^{p+1}\sqrt{\rho_x^{\otimes(p+1)}}\big[\tilde{L}_j^{(r)}, \tilde{L}_k^{(r)}\big]\sqrt{\rho_x^{\otimes(p+1)}}\right\|_1}{p+1} \\
&= \frac{\left\|(1/p)\sum_{q=1}^{p+1}\sum_{r\neq q}\sqrt{\rho_x^{\otimes(p+1)}}\big[\tilde{L}_j^{(r)}, \tilde{L}_k^{(r)}\big]\sqrt{\rho_x^{\otimes(p+1)}}\right\|_1}{p+1} \\
&\leqslant \frac{\sum_{q=1}^{p+1}\left\|\sum_{r\neq q}\sqrt{\rho_x^{\otimes(p+1)}}\big[\tilde{L}_j^{(r)}, \tilde{L}_k^{(r)}\big]\sqrt{\rho_x^{\otimes(p+1)}}\right\|_1}{p(p+1)} \\
&= \frac{(p+1)\left\|\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{jp}, \tilde{L}_{kp}]\sqrt{\rho_x^{\otimes p}}\right\|_1}{p(p+1)} \\
&= \frac{\left\|\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{jp}, \tilde{L}_{kp}]\sqrt{\rho_x^{\otimes p}}\right\|_1}{p},
\end{aligned} \quad (78)$$

i.e., $\frac{(C_{p+1})_{jk}}{p+1} \leqslant \frac{(C_p)_{jk}}{p}$, which implies $\|\frac{C_{p+1}}{p+1}\|_F \leqslant \|\frac{C_p}{p}\|_F$.

A necessary condition for the saturation of the QCRB under the p-local measurement is $\frac{C_p}{p} = 0$, which implies $\frac{\|\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{jp},\tilde{L}_{kp}]\sqrt{\rho_x^{\otimes p}}\|_1}{p} = 0$ for any j, k. This is equivalent to $\frac{\|\sqrt{\rho_x^{\otimes p}}[L_{jp},L_{kp}]\sqrt{\rho_x^{\otimes p}}\|_1}{p} = 0$ for any j, k in the original parametrization, and can be seen as the partial commutative condition under the p-local measurement.

At $p = 1$, the condition $\frac{C_p}{p} = 0$ is equivalent to the partial commutative condition. It is natural to ask whether this condition recovers the weak commutative condition at $p \to \infty$. In the Appendix B we explicitly show that this condition indeed reduces to the weak commutative condition when $p \to \infty$. Specifically we show that (regardless of the parametrization)

$$\lim_{p\to\infty}\frac{\left\|\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{jp}, \tilde{L}_{kp}]\sqrt{\rho_x^{\otimes p}}\right\|_1}{p} = |\text{Tr}(\rho_x[\tilde{L}_j, \tilde{L}_k])|. \quad (79)$$

When $p \to \infty$ the partial commutative condition $\frac{C_p}{p} = 0$ is then equivalent to the weak commutative condition $\tilde{F}_{\text{Im}} = 0$, where $(\tilde{F}_{\text{Im}})_{jk} = \frac{1}{2}\text{Tr}(\rho_x[\tilde{L}_j, \tilde{L}_k])$. This clarifies the connection between the partial commutative condition and the weak commutative condition and solves an open question [23]. The connection also suggests that the partial commutative condition under p-local measurements $\frac{C_p}{p} = 0$ is likely also sufficient for the saturation of QCRB under p-local measurements, although we do not have a proof.

Since $\|\frac{C_p}{p}\|_F$ is monotone, we have

$$\|C_1\|_F \geqslant \left\|\frac{C_2}{2}\right\|_F \geqslant \cdots \geqslant \lim_{p\to\infty}\left\|\frac{C_p}{p}\right\|_F = \|\tilde{F}_{\text{Im}}\|_F, \quad (80)$$

where $(C_1)_{jk} = \frac{1}{2}\|\sqrt{\rho_x}[\tilde{L}_j, \tilde{L}_k]\sqrt{\rho_x}\|_1$ and $(\tilde{F}_{\text{Im}})_{jk} = \frac{1}{2i}\text{Tr}(\rho_x[\tilde{L}_j, \tilde{L}_k])$ with $\tilde{L}_j$ and $\tilde{L}_k$ as the SLDs under the reparametrization that $\tilde{F}_Q = I$. All values of $\frac{(C_p)_{jk}}{p}$ are thus between $\frac{1}{2}|\text{Tr}(\sqrt{\rho_x}[\tilde{L}_j, \tilde{L}_k]\sqrt{\rho_x})|$ and $\frac{1}{2}\|\sqrt{\rho_x}[\tilde{L}_j, \tilde{L}_k]\sqrt{\rho_x}\|_1$, i.e., between the absolute value of the trace and the trace norm of the same matrix $\frac{1}{2}\sqrt{\rho_x}[\tilde{L}_j, \tilde{L}_k]\sqrt{\rho_x}$.

When $p \to \infty$, by substituting $\lim_{p\to\infty}\|\frac{C_p}{p}\|_F = \|\tilde{F}_{\text{Im}}\|_F$ into the bound

$$\Gamma_p \leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2, \quad (81)$$

we have

$$\Gamma_\infty \leqslant n - \frac{1}{4(n-1)}\|\tilde{F}_{\mathrm{Im}}\|_F^2. \tag{82}$$

Combined with the lower bound in Eq. (6) [30], which is

$$\Gamma_\infty \geqslant \frac{n^2}{n+\|\tilde{F}_{\mathrm{Im}}\|_1} \geqslant n - \|\tilde{F}_{\mathrm{Im}}\|_1, \tag{83}$$

we get

$$n - \|\tilde{F}_{\mathrm{Im}}\|_1 \leqslant \Gamma_\infty \leqslant n - \frac{1}{4(n-1)}\|\tilde{F}_{\mathrm{Im}}\|_F^2, \tag{84}$$

where $\tilde{F}_{\mathrm{Im}} = F_Q^{-\frac{1}{2}}F_{\mathrm{Im}}F_Q^{-\frac{1}{2}}$. It can be easily seen that the QCRB is saturable (in which case $\Gamma_\infty = n$) if and only if $\tilde{F}_{\mathrm{Im}} = 0$, which is just the weak commutative condition. This provides an alternative way of showing the weak commutative condition is necessary and sufficient for the saturation of QCRB at $p\to\infty$.

$\tilde{F}_{\mathrm{Im}}$ has been proposed as a measure of quantumness based on the lower bound $\Gamma_\infty \geqslant n - \|\tilde{F}_{\mathrm{Im}}\|_1$ [30]. The upper bound obtained here adds another layer on the interpretation of $\tilde{F}_{\mathrm{Im}}$ as the quantumness when $p\to\infty$. We note that if $\frac{C_p}{p}=0$ is also sufficient for the saturation of the QCRB under p-local measurements, $\frac{C_p}{p}$ can be used as a measure of the quantumness under p-local measurements.

Similar to the case of pure states, we can also obtain the other bounds as

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] \leqslant n - f(n)\left\|\frac{F_Q^{-\frac{1}{2}}\bar{\mathbf{F}}_{\mathrm{Im}p}F_Q^{-\frac{1}{2}}}{p}\right\|_F^2, \tag{85}$$

where $f(n)=\max\{\frac{1}{4(n-1)},\frac{n-2}{(n-1)^2},\frac{1}{5}\}$, $\bar{\mathbf{F}}_{\mathrm{Im}p}$ is the imaginary part of $\bar{\mathbf{F}}=\sum_q \bar{\mathbf{F}}_{u_q}$ with $\bar{\mathbf{F}}_{u_q}$ equal to either F_{u_q} or $F_{u_q}^T$, where F_{u_q} is a $n\times n$ matrix with the jkth entry given by

$$(F_{u_q})_{jk} = \langle u_q|\sqrt{\rho_x^{\otimes p}}L_{jp}L_{kp}\sqrt{\rho_x^{\otimes p}}|u_q\rangle, \tag{86}$$

L_{jp} is the SLD of $\rho_x^{\otimes p}$ corresponding to the parameter x_j, and $\{|u_q\rangle\}$ are a set of vectors in $H_d^{\otimes p}$ that satisfies $\sum_q |u_q\rangle\langle u_q| = I_{d^p}$ with I_{d^p} denote the $d^p\times d^p$ identity matrix.

D. Simpler bounds of the incompatibility measures

The obtained tradeoff relation under the p-local measurement in Eq. (74) needs to compute $\|\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{jp},\tilde{L}_{kp}]\sqrt{\rho_x^{\otimes p}}\|_1$, which involves operators whose dimension increases exponentially with p. Here we provide an alternative tradeoff relation, which only uses operators on a single quantum state thus easier to compute.

If we write $\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{jp},\tilde{L}_{kp}]\sqrt{\rho_x^{\otimes p}} = D_p^{(jk)} + O_p^{(jk)}$ with $D_p^{(jk)}$ as the diagonal part and $O_p^{(jk)}$ as the off-diagonal part, we have (see Appendix B)

$$\|D_p^{(jk)}\|_1 \leqslant \|D_p^{(jk)}+O_p^{(jk)}\|_1 \leqslant \|D_p^{(jk)}\|_1 + \|O_p^{(jk)}\|_1. \tag{87}$$

In the Appendix B we show that with the eigenvalue decomposition $\rho_x=\sum_{q=1}^m \lambda_q|\Psi_q\rangle\langle\Psi_q|$ with $\lambda_q>0$,

$$\|D_p^{(jk)}\|_1 = \sum_{v_1,\dots,v_p}\left(\prod_{r=1}^p \lambda_{v_r}\right)\left|\sum_{r=1}^p \langle\Psi_{v_r}|[\tilde{L}_j,\tilde{L}_k]|\Psi_{v_r}\rangle\right|, \tag{88}$$

where $v_1,\dots,v_p\in\{1,\dots,m\}$, $\tilde{L}_j=\sum_q (F_Q^{-\frac{1}{2}})_{jq}L_q$ and $\tilde{L}_k=\sum_q (F_Q^{-\frac{1}{2}})_{kq}L_q$. As shown in the Appendix B, $\|O_p^{(jk)}\|_1 \approx O(\sqrt{p})$, the difference between $\frac{\|D_p^{(jk)}\|_1}{p}$ and $\frac{\|\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{jp},\tilde{L}_{kp}]\sqrt{\rho_x^{\otimes p}}\|_1}{p}$ is then within the order of $\frac{1}{\sqrt{p}}$, i.e.,

$$\begin{aligned}\frac{\|D_p^{(jk)}\|_1}{p} &\leqslant \frac{\|\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{jp},\tilde{L}_{kp}]\sqrt{\rho_x^{\otimes p}}\|_1}{p}\\ &\leqslant \frac{\|D_p^{(jk)}\|_1}{p} + O\left(\frac{1}{\sqrt{p}}\right).\end{aligned} \tag{89}$$

Here $\|D_p^{(jk)}\|_1$ is quantitatively equivalent to the expected value of $|\sum_{r=1}^p\langle\Psi_{v_r}|[\tilde{L}_j,\tilde{L}_k]|\Psi_{v_r}\rangle|$ with each eigenvector $|\Psi_{v_r}\rangle$ selected independently with probability λ_{v_r}, i.e.,

$$\|D_p^{(jk)}\|_1 = E\left(\left|\sum_{r=1}^p\langle\Psi_{v_r}|[\tilde{L}_j,\tilde{L}_k]|\Psi_{v_r}\rangle\right|\right), \tag{90}$$

where $E(\cdot)$ denotes the expectation, each $|\Psi_{v_r}\rangle$ is randomly and independently chosen from the eigenvectors of ρ_x with a probability of the corresponding eigenvalue λ_{v_r}.

By replacing $\|\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{jp},\tilde{L}_{kp}]\sqrt{\rho_x^{\otimes p}}\|_1$ with $\|D_p^{(jk)}\|_1$, we then obtain an alternative bound

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] \leqslant n - \frac{1}{4(n-1)}\left\|\frac{T_p}{p}\right\|_F^2, \tag{91}$$

with

$$(T_p)_{jk} = \frac{1}{2}E\left(\left|\sum_{r=1}^p\langle\Psi_{v_r}|[\tilde{L}_j,\tilde{L}_k]|\Psi_{v_r}\rangle\right|\right). \tag{92}$$

Here T_p is also monotonically decreasing with p as

$$\|T_1\|_F \geqslant \left\|\frac{T_2}{2}\right\|_F \geqslant \cdots \geqslant \lim_{p\to\infty}\left\|\frac{T_p}{p}\right\|_F = \|\tilde{F}_{\mathrm{Im}}\|_F, \tag{93}$$

where $(\tilde{F}_{\mathrm{Im}})_{jk} = \frac{1}{2}|\mathrm{Tr}(\rho_x[\tilde{L}_j,\tilde{L}_k])|$.

We note that this bound can be equivalently obtained by choosing the set of $\{|u_q\rangle\}$ in Eq. (49) as the eigenvectors of ρ_x instead of the eigenvectors of $\sqrt{\rho_x}[\tilde{L}_j,\tilde{L}_k]\sqrt{\rho_x}$.

V. INCOMPATIBILITY MEASURES WITH RLDs

The approach can be used to obtain various other incompatibility measures with different operators. Here we demonstrate it with the right logarithmic operators (RLD) [1,61].

The quantum Cramer-Rao bound in terms of the RLD quantum Fisher information is given by

$$\mathrm{Cov}(\hat{x}) \geqslant \frac{1}{\nu}(F^{\mathrm{RLD}})^{-1}, \tag{94}$$

where $(F^{\mathrm{RLD}})_{jk} = \mathrm{Tr}(\rho_x L_j^R L_k^{R\dagger})$, L_j^R (L_k^R) is the RLD associated with the parameter x_j (x_k), which can be obtained from the equation $\partial_{x_j}\rho_x = \rho_x L_j^R$ [1,18,61]. Different from the SLD quantum Fisher information matrix, the RLD quantum Fisher information matrix is in general a complex matrix. If we decompose the inverse of the RLD quantum Fisher information matrix into the real and imaginary parts as $(F^{\mathrm{RLD}})^{-1} = (F^{\mathrm{RLD}})^{-1}_{\mathrm{Re}} + i(F^{\mathrm{RLD}})^{-1}_{\mathrm{Im}}$, Eq. (94) then leads to the standard RLD lower bound on the weighted covariance matrix as

$$\nu\,\mathrm{Tr}[W\,\mathrm{Cov}(\hat{x})] \geqslant \mathrm{Tr}\big[W(F^{\mathrm{RLD}})^{-1}_{\mathrm{Re}}\big] + \big\|\sqrt{W}(F^{\mathrm{RLD}})^{-1}_{\mathrm{Im}}\sqrt{W}\big\|_1. \quad (95)$$

For single-parameter estimation the standard RLD bound is always less tighter than the SLD bound. For multiparameter quantum estimation, however, the RLD bound can be tighter than the SLD bound [36,50,61].

We can obtain an upper bound on Γ_p from the standard RLD bound. As $\mathrm{Cov}^{-1}(\hat{x}) \leqslant \nu F^{\mathrm{RLD}}$, by writing $F^{\mathrm{RLD}} = F^{\mathrm{RLD}}_{\mathrm{Re}} + iF^{\mathrm{RLD}}_{\mathrm{Im}}$ as the real and imaginary parts, we have

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] \leqslant \mathrm{Tr}\big[F_Q^{-1}F^{\mathrm{RLD}}_{\mathrm{Re}}\big] - \big\|F_Q^{-\frac{1}{2}}F^{\mathrm{RLD}}_{\mathrm{Im}}F_Q^{-\frac{1}{2}}\big\|_1. \quad (96)$$

This bound is independent of p since the RLD bound holds under any measurements.

We now show how the standard RLD bound can be improved in a similar way. By choosing the operators as $(X_1,\dots,X_n,L_1^{R\dagger},\dots,L_n^{R\dagger})$, we have

$$\begin{aligned} S_u &= \big(X_1\sqrt{\rho_x}|u\rangle\ \dots\ X_n\sqrt{\rho_x}|u\rangle\, L_1^{R\dagger}\sqrt{\rho_x}|u\rangle\ \dots\ L_n^{R\dagger}\sqrt{\rho_x}|u\rangle\big)^\dagger \\ &\quad \big(X_1\sqrt{\rho}|u\rangle\ \dots\ X_n\sqrt{\rho}|u\rangle\, L_1^{R\dagger}\sqrt{\rho}|u\rangle\ \dots\ L_n^{R\dagger}\sqrt{\rho}|u\rangle\big) \\ &= \begin{pmatrix} A_u & B_u \\ B_u^\dagger & F_u \end{pmatrix} \geqslant 0, \end{aligned} \quad (97)$$

with $(A_u)_{jk} = \langle u|\sqrt{\rho_x}X_jX_k\sqrt{\rho_x}|u\rangle$, $(B_u)_{jk} = \langle u|\sqrt{\rho_x}X_jL_k^{R\dagger}\sqrt{\rho_x}|u\rangle$, $(F_u)_{jk} = \langle u|\sqrt{\rho_x}L_j^RL_k^{R\dagger}\sqrt{\rho_x}|u\rangle$.

Similarly, if we choose a set of $\{|u_q\rangle\}$ with $\sum_q |u_q\rangle\langle u_q| = I$, we can get $\bar{\mathbf{S}} = \sum_q \bar{\mathbf{S}}_{u_q}$ with $\bar{\mathbf{S}}_{u_q} \in \{S_{u_q}, S_{u_q}^T\}$. The standard RLD bound corresponds to choosing $\bar{\mathbf{S}}_{u_q} = S_{u_q}$ for all q. In this case $\bar{\mathbf{S}} = \sum_q S_{u_q} = \begin{pmatrix} A & B \\ B^\dagger & F^{\mathrm{RLD}}\end{pmatrix} \geqslant 0$, where $(A)_{jk} = \mathrm{Tr}(\rho_xX_jX_k)$, $(B)_{jk} = \mathrm{Tr}(\rho_xX_jL_k^{R\dagger})$, $(F^{\mathrm{RLD}})_{jk} = \mathrm{Tr}(\rho_xL_j^RL_k^{R\dagger})$. From the local unbiased condition,

$$\mathrm{Tr}\big(\rho_xL_j^R\hat{X}_k\big) = \delta_j^k, \quad (98)$$

we can get

$$\begin{aligned} (B)_{jk} &= \mathrm{Tr}\big(\rho_xX_jL_k^{R\dagger}\big) \\ &= \mathrm{Tr}\big(\rho_xL_k^RX_j\big)^* \\ &= \delta_k^j, \end{aligned} \quad (99)$$

thus, in this case $B = I$. The standard RLD bound can then be obtained via the Schur's complement as

$$\mathrm{Cov}(\hat{x}) \geqslant A \geqslant B(F^{\mathrm{RLD}})^{-1}B^\dagger = (F^{\mathrm{RLD}})^{-1}. \quad (100)$$

If it is repeated with ν times, we then obtain the standard RLD bound

$$\mathrm{Cov}(\hat{x}) \geqslant \frac{1}{\nu}(F^{\mathrm{RLD}})^{-1}, \quad (101)$$

which then leads to the upper bound on Γ_p as in Eq. (96). For any p-local measurements, we can replace ρ_x with $\rho_x^{\otimes p}$ and repeat the measurement ν/p times, which leads to the same tradeoff relation as in Eq. (96). This is consistent with the fact the standard RLD bound holds for any measurements.

The standard RLD bound can be improved by making proper choices on $\{|u_q\rangle\}$ and $\{\bar{\mathbf{S}}_{u_j}\}$. Here we make a particular choice as an illustration. Again, we first assume $F_Q = I$ and for a fixed pair of indices j, k, choose a complete basis $\{|u_1\rangle,\dots,|u_d\rangle\}$, as the orthonormal eigenvectors of $\sqrt{\rho_x}(L_j^RL_k^{R\dagger} - L_k^RL_j^{R\dagger})\sqrt{\rho_x}$. For any $|u_q\rangle$, the imaginary part of $(F_{u_q})_{jk}$ is $\frac{1}{2i}\langle u_q|\sqrt{\rho_x}(L_j^RL_k^{R\dagger} - L_k^RL_j^{R\dagger})\sqrt{\rho_x}|u_q\rangle$, which we denote as t_{jk}^q. We then let

$$\bar{\mathbf{S}}_{u_q} := \begin{cases} S_{u_q} & \text{when} \quad t_{jk}^q \geqslant 0, \\ S_{u_q}^T & \text{when} \quad t_{jk}^q < 0. \end{cases} \quad (102)$$

In this case we get

$$\bar{\mathbf{S}} = \sum_q \bar{\mathbf{S}}_{u_q} = \begin{pmatrix} \bar{\mathbf{A}} & \bar{\mathbf{B}} \\ \bar{\mathbf{B}}^\dagger & \bar{\mathbf{F}}^{\mathrm{RLD}} \end{pmatrix}, \quad (103)$$

where $\bar{\mathbf{B}} = I + i\bar{\mathbf{B}}_{\mathrm{Im}}$, $\bar{\mathbf{F}}^{\mathrm{RLD}} = \sum_q \bar{\mathbf{F}}_{u_q}$ with $\bar{\mathbf{F}}_{u_q}$ equals to either F_{u_q} or $F_{u_q}^T$ according to the choices in Eq. (102) [which makes the imaginary part of $(\bar{\mathbf{F}}_{u_q})_{jk}$ always positive]. The real part of $\bar{\mathbf{F}}^{\mathrm{RLD}}$ remains the same as $F^{\mathrm{RLD}}_{\mathrm{Re}}$, the imaginary part of the jkth entry of $\bar{\mathbf{F}}^{\mathrm{RLD}}$ is

$$\big(\bar{\mathbf{F}}^{\mathrm{RLD}}_{\mathrm{Im}}\big)_{jk} = \tfrac{1}{2}\big\|\sqrt{\rho_x}\big(L_j^RL_k^{R\dagger} - L_k^RL_j^{R\dagger}\big)\sqrt{\rho_x}\big\|_1. \quad (104)$$

By following the same procedure, we can obtain the tradeoff relation under the 1-local measurement (under the parametrization such that $F_Q = I$) as

$$\mathrm{Tr}[\mathrm{Cov}^{-1}(\hat{x})] \leqslant \mathrm{Tr}\big[\bar{\mathbf{F}}^{\mathrm{RLD}}_{\mathrm{Re}}\big] - \frac{1}{4(n-1)}\big\|C_1^{\mathrm{RLD}}\big\|_F^2, \quad (105)$$

where $(C_1^{\mathrm{RLD}})_{jk} = \min\{\frac{1}{2}\|\sqrt{\rho_x}(L_j^RL_k^{R\dagger} - L_k^RL_j^{R\dagger})\sqrt{\rho_x}\|_1, 2\}$ (see Appendix E). If we repeat the 1-local measurement on ν copies of the state, the tradeoff relation under 1-local measurements, with the parametrization such that $F_Q = I$, is then

$$\frac{1}{\nu}\mathrm{Tr}[\mathrm{Cov}^{-1}(\hat{x})] \leqslant \mathrm{Tr}\big[F^{\mathrm{RLD}}_{\mathrm{Re}}\big] - \frac{1}{4(n-1)}\big\|C_1^{\mathrm{RLD}}\big\|_F^2. \quad (106)$$

When $F_Q \neq I$ initially, we can first make a reparametrization with $\tilde{x} = F_Q^{-\frac{1}{2}}x$. The tradeoff relation in Eq. (106) can then be expressed in the original parametrization as

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] \leqslant \mathrm{Tr}\big[F_Q^{-1}F^{\mathrm{RLD}}_{\mathrm{Re}}\big] - \frac{1}{4(n-1)}\big\|C_1^{\mathrm{RLD}}\big\|_F^2 \quad (107)$$

with the entries of C_1^{RLD} given by $(C_1^{\mathrm{RLD}})_{jk} = \min\{\frac{1}{2}\|\sqrt{\rho_x}(\tilde{L}_j^R\tilde{L}_k^{R\dagger} - \tilde{L}_k^R\tilde{L}_j^{R\dagger})\sqrt{\rho_x}\|_1, 2\}$, where $\tilde{L}_j^R =$

$\sum_q (F_Q^{-\frac{1}{2}})_{jq} L_q^R$ and $\tilde{L}_k^R = \sum_q (F_Q^{-\frac{1}{2}})_{kq} L_q^R$ (see Appendix E for detail).

For p-local measurements, we can similarly get

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] \leqslant \mathrm{Tr}\big[F_Q^{-1}F_{\mathrm{Re}}^{\mathrm{RLD}}\big] - \frac{1}{4(n-1)}\left\|\frac{C_p^{\mathrm{RLD}}}{p}\right\|_F^2, \tag{108}$$

where $(C_p^{\mathrm{RLD}})_{jk} = \min\{\frac{1}{2}\|\sqrt{\rho_x^{\otimes p}}(\tilde{L}_{jp}^R\tilde{L}_{kp}^{R\dagger} - \tilde{L}_{kp}^R\tilde{L}_{jp}^{R\dagger})\sqrt{\rho_x^{\otimes p}}\|_1, 2p\}$.

VI. EXAMPLES

A. Example 1

Consider a state $\rho_x = \frac{1}{2}(I + \delta\sigma_3 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)$, where the true values of the parameters x_1, x_2, x_3 are all 0 and $|\delta| < 1$. The eigenvectors of ρ_x are $|0\rangle$ and $|1\rangle$ with $\rho_x|0\rangle = \frac{1}{2}(1+\delta)|0\rangle$, $\rho_x|1\rangle = \frac{1}{2}(1-\delta)|1\rangle$. The SLD operators corresponding to the parameters can be easily obtained as

$$L_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} \frac{1}{1+\delta} & 0 \\ 0 & \frac{-1}{1-\delta} \end{pmatrix}, \tag{109}$$

from which we can get the QFIM

$$F_Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{1-\delta^2} \end{pmatrix}. \tag{110}$$

The SLD under the reparametrization $\tilde{x} = F_Q^{\frac{1}{2}} x$ are given by

$$\tilde{L}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{L}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tilde{L}_3 = \begin{pmatrix} \sqrt{\frac{1-\delta}{1+\delta}} & 0 \\ 0 & -\sqrt{\frac{1+\delta}{1-\delta}} \end{pmatrix}. \tag{111}$$

From $(C_1)_{jk} = \frac{1}{2}\|\sqrt{\rho_x}[\tilde{L}_j, \tilde{L}_k]\sqrt{\rho_x}\|_1$ we have

$$C_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \tag{112}$$

which gives the tradeoff relation under the 1-local measurement as

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \leqslant n - \frac{1}{4(n-1)}\|C_1\|_F^2 = \frac{9}{4}. \tag{113}$$

with

$$(T_1)_{jk} = \frac{1}{2}\left\{\frac{1+\delta}{2}|\langle 0|[\tilde{L}_j, \tilde{L}_k]|0\rangle| + \frac{1-\delta}{2}|\langle 1|[\tilde{L}_j, \tilde{L}_k]|1\rangle|\right\}, \tag{114}$$

we can obtain the bound with T_1 as

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \leqslant n - \frac{1}{4(n-1)}\|T_1\|_F^2 = \frac{11}{4}. \tag{115}$$

If we choose a set of $\{|u_q\rangle\}$ as $|u_0\rangle = \binom{1}{0}$, $|u_1\rangle = \binom{0}{1}$, which satisfies $|u_0\rangle\langle u_0| + |u_1\rangle\langle u_1| = I$, from $(F_{u_q})_{jk} = \langle u_q|\sqrt{\rho_x}\tilde{L}_j\tilde{L}_k\sqrt{\rho_x}|u_q\rangle$ we can obtain

$$F_{u_0} = \frac{1}{2}\begin{pmatrix} 1+\delta & i(1+\delta) & 0 \\ -i(1+\delta) & 1+\delta & 0 \\ 0 & 0 & 1-\delta \end{pmatrix}, \quad F_{u_1} = \frac{1}{2}\begin{pmatrix} 1-\delta & -i(1-\delta) & 0 \\ i(1-\delta) & 1-\delta & 0 \\ 0 & 0 & 1+\delta \end{pmatrix}. \tag{116}$$

We can choose $\bar{\mathbf{F}} = F_{u_0} + F_{u_1}^T$ whose imaginary part is

$$\bar{\mathbf{F}}_{\mathrm{Im}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{117}$$

the tradeoff relation in Eq. (85) then gives

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \leqslant n - \frac{(n-2)}{(n-1)^2}\|\bar{\mathbf{F}}_{\mathrm{Im}}\|_F^2 = \frac{5}{2}. \tag{118}$$

For 2-local measurement, using $(C_2)_{jk} = \frac{1}{2}\|\sqrt{\rho_x^{\otimes 2}}[\tilde{L}_{j2}, \tilde{L}_{k2}]\sqrt{\rho_x^{\otimes 2}}\|_1$ with $\tilde{L}_{j2} = \tilde{L}_j \otimes I + I \otimes \tilde{L}_j$, we can obtain

$$C_2 = \begin{pmatrix} 0 & 1+\delta^2 & \sqrt{1+\delta^2} \\ 1+\delta^2 & 0 & \sqrt{1+\delta^2} \\ \sqrt{1+\delta^2} & \sqrt{1+\delta^2} & 0 \end{pmatrix}, \tag{119}$$

which gives the tradeoff relation

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_2}{2}\right\|_F^2 = \frac{45}{16} - \frac{1}{4}\delta^2 - \frac{1}{16}\delta^4. \tag{120}$$

From

$$\begin{aligned}(T_2)_{jk} =& \frac{1}{2}\left(\frac{1+\delta}{2}\right)^2 |\langle 0|[\tilde{L}_j, \tilde{L}_k]|0\rangle + \langle 0|[\tilde{L}_j, \tilde{L}_k]|0\rangle| \\ &+ \frac{1+\delta}{2}\frac{1-\delta}{2}|\langle 0|[\tilde{L}_j, \tilde{L}_k]|0\rangle + \langle 1|[\tilde{L}_j, \tilde{L}_k]|1\rangle| \\ &+ \frac{1}{2}\left(\frac{1-\delta}{2}\right)^2 |\langle 1|[\tilde{L}_j, \tilde{L}_k]|1\rangle + \langle 1|[\tilde{L}_j, \tilde{L}_k]|1\rangle|,\end{aligned} \tag{121}$$

we have

$$T_2 = \begin{pmatrix} 0 & 1+\delta^2 & 0 \\ 1+\delta^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{122}$$

which gives the tradeoff relation with T_2 as

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \leqslant n - \frac{1}{4(n-1)}\left\|\frac{T_2}{2}\right\|_F^2 = \frac{47}{16} - \frac{\delta^2}{8} - \frac{\delta^4}{16}. \tag{123}$$

If we choose a set of $\{|u_q\rangle\}$ in the two-qubit space as $|u_0\rangle = |00\rangle$, $|u_1\rangle = |01\rangle$, $|u_2\rangle = |10\rangle$, $|u_3\rangle = |11\rangle$, we can obtain

$$F_{u_0} = \frac{1}{2}\begin{pmatrix} (1+\delta)^2 & i(1+\delta)^2 & 0 \\ -i(1+\delta)^2 & (1+\delta)^2 & 0 \\ 0 & 0 & 2(1-\delta^2) \end{pmatrix},$$

$$F_{u_1} = \frac{1}{2}\begin{pmatrix} 1-\delta^2 & 0 & 0 \\ 0 & 1-\delta^2 & 0 \\ 0 & 0 & 2\delta^2 \end{pmatrix},$$

$$F_{u_2} = \frac{1}{2}\begin{pmatrix} 1-\delta^2 & 0 & 0 \\ 0 & 1-\delta^2 & 0 \\ 0 & 0 & 2\delta^2 \end{pmatrix},$$

$$F_{u_3} = \frac{1}{2}\begin{pmatrix} (1-\delta)^2 & -i(1-\delta)^2 & 0 \\ i(1-\delta)^2 & (1-\delta)^2 & 0 \\ 0 & 0 & 2(1-\delta^2) \end{pmatrix}, \quad (124)$$

where the entries of F_{u_q} are obtained as $(F_{u_q})_{jk} = \langle u_q|\sqrt{\rho_x^{\otimes 2}}\tilde{L}_{j2}\tilde{L}_{k2}\sqrt{\rho_x^{\otimes 2}}|u_q\rangle$. Let $\bar{\mathbf{F}} = F_{u_0} + F_{u_1} + F_{u_2} + F_{u_3}^T$, which has the imaginary part as

$$\bar{\mathbf{F}}_{\mathrm{Im}2} = \begin{pmatrix} 0 & 1+\delta^2 & 0 \\ -(1+\delta^2) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (125)$$

This gives the tradeoff relation

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \leqslant n - \frac{(n-2)}{(n-1)^2}\left\|\frac{\bar{\mathbf{F}}_{\mathrm{Im}2}}{2}\right\|_F^2 = 3 - \frac{1}{8}(1+\delta^2)^2. \quad (126)$$

When $\delta = 0$, i.e., $\rho_x = \frac{1}{2}(I + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)$, the tradeoff relations can be analytically calculated under general p-local measurement. In this case the SLD operators under the reparametrization are given by $\tilde{L}_1 = \sigma_1, \tilde{L}_2 = \sigma_2, \tilde{L}_3 = \sigma_3$, thus,

$$(C_p)_{12} = \frac{1}{2}\left\|\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{1p}, \tilde{L}_{2p}]\sqrt{\rho_x^{\otimes p}}\right\|_1 = \frac{1}{2}\left\|\sqrt{\rho_x^{\otimes p}}[\sigma_{1p}, \sigma_{2p}]\sqrt{\rho_x^{\otimes p}}\right\|_1 = \frac{1}{2^p}\|\sigma_{3p}\|_1, \quad (127)$$

where $\sigma_{lp} = \sum_{r=1}^{p}\sigma_l^{(r)}$ for $l \in \{1, 2, 3\}$. As the eigenvalues of σ_{lp} are $-p + 2s$ with multiplicity $\binom{p}{s}$, where $s = 0, 1, \ldots, p$, thus,

$$\|\sigma_{3p}\|_1 = \sum_{s=0}^{p}\binom{p}{s}|-p+2s| = 2\sum_{s=0}^{\lfloor\frac{p}{2}\rfloor}\binom{p}{s}(p-2s) = \begin{cases} 2p\binom{p-1}{\frac{p-1}{2}}, & \text{if } p \text{ is odd} \\ p\binom{p}{\frac{p}{2}}, & \text{if } p \text{ is even.} \end{cases} \quad (128)$$

Due to the symmetry, $(C_p)_{jk}$ takes the same value for all $j \neq k \in \{1, 2, 3\}$. The tradeoff relation under the p-local measurement is then given by

$$\Gamma_p = \frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2 = 3 - \frac{3}{4}\left(\frac{\mathcal{N}_p}{p}\right)^2, \quad (129)$$

where $\mathcal{N}_p = \frac{1}{2^p}\|\sigma_{3p}\|_1$.

For the bound with T_p, we have

$$(T_p)_{12} = \frac{1}{2}\sum_{s=0}^{p}\binom{p}{s}\left(\frac{1+\delta}{2}\right)^s\left(\frac{1-\delta}{2}\right)^{p-s}|2s - 2(p-s)| = \frac{1}{2^p}\sum_{s=0}^{p}\binom{p}{s}(1+\delta)^s(1-\delta)^{p-s}|2s-p|, \quad (130)$$

and $(T_p)_{13} = (T_p)_{23} = 0$, thus,

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] \leqslant n - \frac{1}{4(n-1)}\left\|\frac{T_p}{p}\right\|_F^2 = 3 - \frac{1}{4p^2}(T_p)_{12}^2. \quad (131)$$

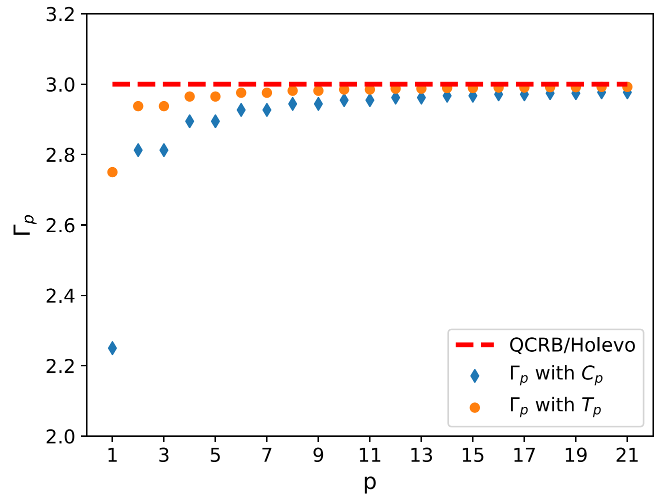

FIG. 1. Upper bounds on Γ_p obtained with C_p and T_p, together with the QCRB and Holevo bounds at the case $\delta = 0$.

In Fig. 1 we plot the bounds as a function of p in the case of $\delta = 0$. Note that in this case the weak commutative condition holds, the Holevo bound equals to the QCRB, which is achievable when $p \to \infty$. For any finite p, however, the bounds are strictly less than 3, thus any collective measurement on finite copies can not saturate the Holevo bound. It can also be seen that the difference between the bounds obtained from C_p and T_p is large for small p, but the difference decreases with p.

We also plot the bounds for the state $\rho_x = \frac{1}{2}(I + \delta\sigma_3 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)$ with general δ in Fig. 2. The complexity of calculating the bound with C_p, which we compute up to $p = 10$, increases exponentially with p. As a comparison, the bound with T_p is much easier to compute, which we compute up to $p = 100$. Since the difference between these two bounds decreases with p, a good strategy is to use the bound with C_p for small p and use the bound with T_p for large p. We also plot

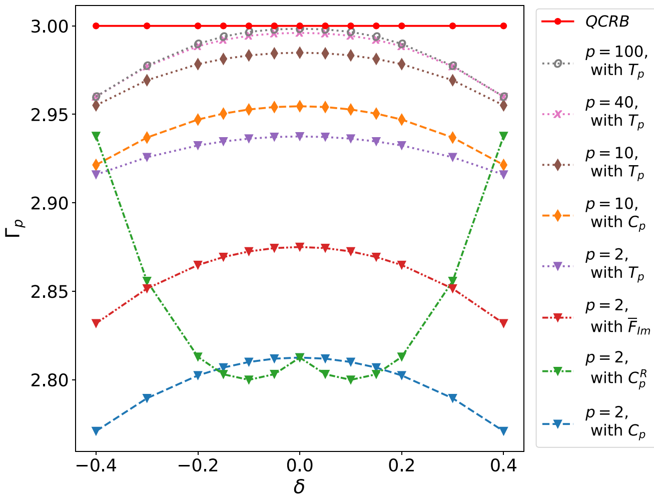

FIG. 2. Comparison of different bounds of Γ_p for the estimation of $\rho_x = \frac{1}{2}(I + \delta\sigma_3 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)$ at $x_1 = x_2 = x_3 = 0$.

the bound with the RLD for $p = 2$, it can be seen that the RLD bound can be either tighter or less tight than the bound with C_p. We can combine these bounds and choose the minimal of them to get a tighter bound.

B. Example 2

We consider another example with a three-dimensional state $\rho_x = \frac{1}{3}I + \sum_j x_j G_j$, where $G_j = \frac{1}{2}\Lambda_j$, where $\{\Lambda_j\}_{j=1}^8$ are the Gell-Mann matrices

$$\Lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\Lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \Lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\Lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \Lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad (132)$$

which form a basis for 3×3 Hermitian matrices. When the true values of the parameters are all 0, the SLDs can be obtained as $L_j = 3G_j$, and $F_Q = \frac{3}{2}I$. The SLDs after the reparametrization which makes $\tilde{F}_Q = I$ are given by $\tilde{L}_j = \sqrt{\frac{2}{3}}L_j = \sqrt{6}G_j$. Since

$$(C_1)_{jk} = \tfrac{1}{2}\|\sqrt{\rho_x}[\tilde{L}_j, \tilde{L}_k]\sqrt{\rho_x}\|_1 = \|[G_j, G_k]\|_1, \quad (133)$$

we have

$$C_1 = \begin{pmatrix} 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}. \quad (134)$$

This gives the tradeoff relation under the 1-local measurement as

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] \leqslant n - \frac{1}{4(n-1)}\|C_1\|_F^2$$
$$= \frac{50}{7} \approx 7.14. \quad (135)$$

For p-local measurement, we can similarly obtain

$$(C_p)_{jk} = \tfrac{1}{2}\Big\|\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{jp}, \tilde{L}_{kp}]\sqrt{\rho_x^{\otimes p}}\Big\|_1 = \frac{1}{3^{p-1}}\|[G_{jp}, G_{kp}]\|_1, \quad (136)$$

where $[G_{jp}, G_{kp}] = \sum_{r=1}^p [G_j^{(r)}, G_k^{(r)}]$. Since the eigenvalues of $[G_j, G_k]$ are $\{-\lambda, 0, \lambda\}$, where $\lambda = \frac{1}{2}(C_1)_{jk}$, the eigenvalues of $[G_{jp}, G_{kp}]$ are given by λs with multiplicity $\binom{p}{s}_2$, for $s = -p, -p+1, \ldots, p$, where $\binom{p}{s}_2 = \sum_{i=0}^p (-1)^i \binom{p}{i}\binom{2p-2i}{p-s-i}$ is the trinomial coefficient, which can be obtained as the $(j+p)$th coefficient of the polynomial $(1 + x + x^2)^p$ (see Appendix F for details). We thus have

$$\|[G_{jp}, G_{kp}]\|_1 = \sum_{s=-p}^p |\lambda s|\binom{p}{s}_2 = 2\lambda \sum_{s=0}^p s\binom{p}{s}_2$$
$$= (C_1)_{jk}\sum_{s=0}^p s\binom{p}{s}_2, \quad (137)$$

where we have used the fact that $\binom{p}{s}_2 = \binom{p}{-s}_2$. Denote $\mathcal{N}_p = \sum_{s=0}^p s\binom{p}{s}_2$, we then have

$$(C_p)_{jk} = \frac{1}{3^{p-1}}\|[G_{jp}, G_{kp}]\|_1 = (C_1)_{jk}\frac{\mathcal{N}_p}{3^{p-1}}, \quad (138)$$

which gives the Frobenius norm of C_p as

$$\|C_p\|_F = \sqrt{\sum_{jk}(C_p)_{jk}^2} = \sqrt{\sum_{jk}\left((C_1)_{jk}\frac{1}{3^{p-1}}\mathcal{N}_p\right)^2}$$
$$= \frac{1}{3^{p-1}}\mathcal{N}_p\sqrt{\sum_{jk}[(C_1)_{jk}]^2} = \frac{1}{3^{p-1}}\mathcal{N}_p\|C_1\|_F. \quad (139)$$

The tradeoff relation under the p-local measurement is then given by

$$\begin{aligned}\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\right] &\leqslant n-\frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2\\ &= n-\frac{1}{4(n-1)}\|C_1\|_F^2\left(\frac{1}{p3^{p-1}}\mathcal{N}_p\right)^2\\ &= 8-\frac{6}{7}\left(\frac{1}{p3^{p-1}}\mathcal{N}_p\right)^2. \end{aligned}\tag{140}$$

Here $\frac{1}{p3^{p-1}}\mathcal{N}_p$ monotonically decreases with p and it is only equal to 0 when $p\to\infty$. The Holevo bound, which equals to the QCRB in this case since the weak commutative condition holds, can thus only be achieved with collective measurement on genuinely infinite number of quantum states in this case.

If there are only three parameters, for example, $\{x_1, x_2, x_5\}$, the associated matrices are given by the 3×3 submatrices of the original ones. Under the 1-local measurement we have

$$C_1=\begin{pmatrix}0 & 1 & \frac{1}{2}\\ 1 & 0 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix},\tag{141}$$

which gives the tradeoff relation

$$\begin{aligned}\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\right] &\leqslant n-\frac{1}{4(n-1)}\|C_1\|_F^2\\ &= 3-\frac{3}{8}=2.625. \end{aligned}\tag{142}$$

Under the p-local measurement, we have

$$\begin{aligned}\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\right] &\leqslant n-\frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2\\ &= n-\frac{1}{4(n-1)}\|C_1\|_F^2\left(\frac{1}{p3^{p-1}}\mathcal{N}_p\right)^2\\ &= 3-\frac{3}{8}\left(\frac{1}{p3^{p-1}}\mathcal{N}_p\right)^2. \end{aligned}\tag{143}$$

For $p=2$, this gives

$$\begin{aligned}\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\right] &\leqslant n-\frac{1}{4(n-1)}\left\|\frac{C_2}{2}\right\|_F^2\\ &= 3-\frac{3}{8}\times\frac{1}{4}\times\frac{16}{9}\\ &= \frac{17}{6}\approx 2.83. \end{aligned}\tag{144}$$

The bound with T_p can be similarly calculated as $(T_p)_{jk}=\frac{1}{2}\sum_{s=0}^{p}\sum_{r=0}^{p-s}\binom{p}{s}\binom{p-s}{r}(\frac{1+3\delta}{3})^s(\frac{1-3\delta}{3})^r(\frac{1}{3})^{p-s-r}|s\times\langle 0|[\tilde{L}_j,\tilde{L}_k]|0\rangle + r\times\langle 1|[\tilde{L}_j,\tilde{L}_k]|1\rangle + (p-s-r)\times\langle 2|[\tilde{L}_j,\tilde{L}_k]|2\rangle|$. For $\delta=0$, the equation can be simplified as

$$(T_p)_{12}=\frac{1}{2}\left(\frac{1}{3}\right)^p\sum_{s=0}^{p}\sum_{r=0}^{p-s}\binom{p}{s}\binom{p-s}{r}|3s-3r|\tag{145}$$

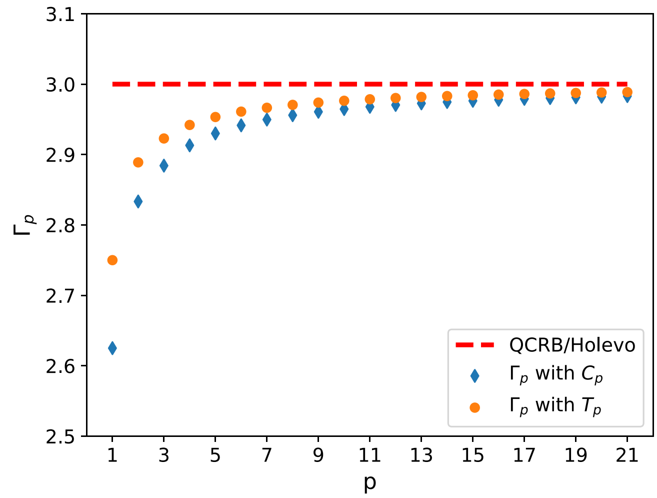

FIG. 3. Precision bounds Γ_p for p-local measurements and the Holevo bound when $n=3$.

and $(T_p)_{13}=0$, $(T_p)_{23}=0$. This then gives

$$\begin{aligned}\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\right] &\leqslant n-\frac{1}{4(n-1)}\left\|\frac{T_p}{p}\right\|_F^2\\ &= 3-\frac{1}{4p^2}(T_p)_{12}^2. \end{aligned}\tag{146}$$

We plot the bounds for $n=3$ as a typical case in Fig. 3. It can be seen that the Holevo bound, which equals to the QCRB as the weak commutative condition holds, is only achievable when $p\to\infty$. For any finite p, the bounds are strictly less than n.

If there are only two parameters, the associated matrices are then given by the 2×2 submatrices of the original ones. For example, suppose the two parameters are $\{x_1, x_2\}$, we then have

$$C_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\tag{147}$$

and the tradeoff relation under the 1-local measurement is then given by

$$\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\right]\leqslant n-\frac{1}{4(n-1)}\|C_1\|_F^2=\frac{3}{2};\tag{148}$$

in this case it is tighter than the Gill-Massar bound.

Under general p-local measurement, we have

$$\begin{aligned}\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\right] &\leqslant n-\frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2\\ &= n-\frac{1}{4(n-1)}\|C_1\|_F^2\left(\frac{1}{p3^{p-1}}\mathcal{N}_p\right)^2\\ &= 2-\frac{1}{2}\left(\frac{1}{p3^{p-1}}\mathcal{N}_p\right)^2. \end{aligned}\tag{149}$$

For $p=2$, we have

$$\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\right]\leqslant\frac{16}{9}\approx 1.78,\tag{150}$$

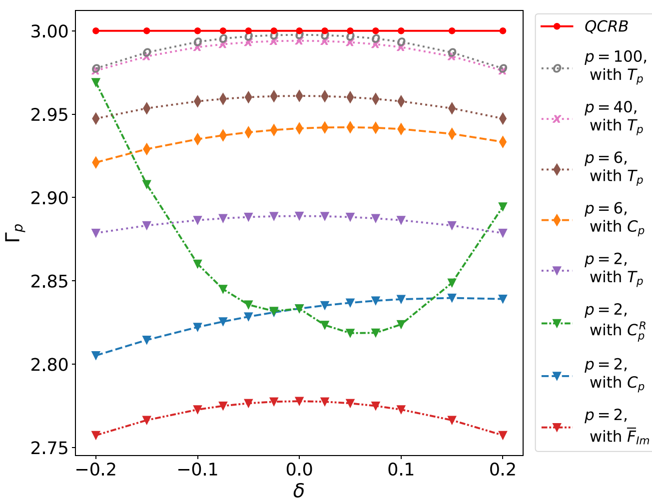

FIG. 4. Comparison of different bounds of Γ_p for the estimation of $\rho_x = \frac{1}{3}I + \delta G_3 + x_1G_1 + x_2G_2 + x_5G_5$ at $x_1 = x_2 = x_5 = 0$.

and for $p = 3$,

$$\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\right] \leqslant \frac{299}{162} \approx 1.85. \tag{151}$$

Similar as the previous example, we also consider the estimation of the state $\rho_x = \frac{1}{3}I + \delta G_3 + x_1G_1 + x_2G_2 + x_5G_5$ with general δ and plot the precision bounds in Fig. 4, where we plotted the bounds with C_p up to $p = 6$ and the bounds with T_p up to $p = 100$. We also plotted the bounds with RLDs and $\bar{\mathbf{F}}_{Im}$ for $p = 2$ (see Appendix F for detailed calculations), as it can be seen the bound given by $\frac{1}{\nu}\mathrm{Tr}[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}] \leqslant n - \frac{(n-2)}{(n-1)^2}\|\frac{\bar{F}_{Im2}}{2}\|_F^2$ is tighter than the bounds given by C_2 and T_2 in this case.

VII. SUMMARY

The presented framework provided a versatile tool to obtain bounds on the precision limit in multiparameter quantum estimation under general p-local measurements, which significantly increased our knowledge on the incompatibility in multiparameter quantum estimation. The relation between the partial commutative condition and the weak commutative condition is also clarified. Future studies include improving the bounds by exploring different choices of $\{|u_q\rangle\}$ and operators in $\bar{\mathbf{S}}$, clarifying whether the partial commutative condition is sufficient for the saturation of the QCRB, and identifying the ultimate precision under general p-local measurements. The approach can also be used to strengthen the uncertainty relations for multiple observables, which is another interesting direction to pursue.

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APPENDIX A: TRADEOFF RELATIONS

We derive the tradeoff relation from

$$S = \begin{pmatrix} A & B \\ B^\dagger & F \end{pmatrix} \geqslant 0, \tag{A1}$$

where A, B, F are $n \times n$ matrices with $\mathrm{Cov}(\hat{x}) \geqslant A$, $B = I + iB_{\mathrm{Im}}$, and $F = F_Q + iF_{\mathrm{Im}}$. We note that the derivation below works regardless whether S is obtained from pure states or mixed states.

Since $\mathrm{Cov}(\hat{x}) \geqslant A$, we have

$$\begin{pmatrix} \mathrm{Cov}(\hat{x}) & B \\ B^\dagger & F \end{pmatrix} = \begin{pmatrix} \mathrm{Cov}(\hat{x}) - A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A & B \\ B^\dagger & F \end{pmatrix} \geqslant 0. \tag{A2}$$

This implies that $F - B^\dagger\mathrm{Cov}^{-1}(\hat{x})B \geqslant 0$. Since $F = F_Q + iF_{\mathrm{Im}}$, $B = I + iB_{\mathrm{Im}}$, we thus have

$$\begin{aligned} &F_Q + iF_{\mathrm{Im}} - \big[\mathrm{Cov}^{-1}(\hat{x}) + B_{\mathrm{Im}}^T\mathrm{Cov}^{-1}(\hat{x})B_{\mathrm{Im}} \\ &\quad + i\big(B_{\mathrm{Im}}^T\mathrm{Cov}^{-1}(\hat{x}) - \mathrm{Cov}^{-1}(\hat{x})B_{\mathrm{Im}}\big)\big] \geqslant 0, \end{aligned} \tag{A3}$$

which implies the real part is positive semidefinite, i.e.,

$$F_Q - \mathrm{Cov}^{-1}(\hat{x}) - B_{\mathrm{Im}}^T\mathrm{Cov}^{-1}(\hat{x})B_{\mathrm{Im}} \geqslant 0. \tag{A4}$$

This can be written as $F_Q - \mathrm{Cov}^{-1}(\hat{x}) \geqslant B_{\mathrm{Im}}^T\mathrm{Cov}^{-1}(\hat{x})B_{\mathrm{Im}} \geqslant 0$, which is stronger than the quantum Cramer-Rao bound $F_Q - \mathrm{Cov}^{-1}(\hat{x}) \geqslant 0$, typically written as $\mathrm{Cov}(\hat{x}) \geqslant F_Q^{-1}$. To saturate the bound, i.e., $\mathrm{Cov}(\hat{x}) = F_Q^{-1}$, we need to have $B_{\mathrm{Im}}^T\mathrm{Cov}^{-1}(\hat{x})B_{\mathrm{Im}} = 0$. When the covariance matrix is full rank, which is always the case when F_Q is invertible, this requires $B_{\mathrm{Im}} = 0$, Eq. (A3) then becomes

$$F_Q + iF_{\mathrm{Im}} - \mathrm{Cov}^{-1}(\hat{x}) \geqslant 0. \tag{A5}$$

The saturation of the quantum Cramer-Rao bound then requires $iF_{\mathrm{Im}} \geqslant 0$. Since F_{Im} is antisymmetric and its eigenvalues are in the form of $\pm i\beta$ with $\beta \in \mathbb{R}$, $iF_{\mathrm{Im}} \geqslant 0$ is only possible when all the eigenvalues are zero, i.e., $F_{\mathrm{Im}} = 0$.

When $F_{\mathrm{Im}} \neq 0$, the QCRB is not saturable. Denote $F_C = \mathrm{Cov}^{-1}(\hat{x})$, and we write Eq. (A3) as

$$F_Q - F_C - B_{\mathrm{Im}}^T F_C B_{\mathrm{Im}} + i\big(F_{\mathrm{Im}} + B_{\mathrm{Im}}^T F_C - F_C B_{\mathrm{Im}}\big) \geqslant 0. \tag{A6}$$

By multiplying $F_Q^{-\frac{1}{2}}$ from both the left and the right, we get

$$\begin{aligned} &I - F_Q^{-\frac{1}{2}}F_C F_Q^{-\frac{1}{2}} - F_Q^{-\frac{1}{2}}B_{\mathrm{Im}}^T F_C B_{\mathrm{Im}} F_Q^{-\frac{1}{2}} \\ &\quad + i\big(F_Q^{-\frac{1}{2}}F_{\mathrm{Im}}F_Q^{-\frac{1}{2}} + F_Q^{-\frac{1}{2}}B_{\mathrm{Im}}^T F_C F_Q^{-\frac{1}{2}} \\ &\quad - F_Q^{-\frac{1}{2}}F_C B_{\mathrm{Im}} F_Q^{-\frac{1}{2}}\big) \geqslant 0. \end{aligned} \tag{A7}$$

Denote $\tilde{F}_C = F_Q^{-\frac{1}{2}}F_C F_Q^{-\frac{1}{2}}$, $\tilde{B}_{\mathrm{Im}} = F_Q^{\frac{1}{2}}B_{\mathrm{Im}}F_Q^{-\frac{1}{2}}$, $\tilde{F}_{\mathrm{Im}} = F_Q^{-\frac{1}{2}}F_{\mathrm{Im}}F_Q^{-\frac{1}{2}}$, we can write the inequality as

$$I - \tilde{F}_C - \tilde{B}_{\mathrm{Im}}^T\tilde{F}_C\tilde{B}_{\mathrm{Im}} + i\big(\tilde{F}_{\mathrm{Im}} + \tilde{B}_{\mathrm{Im}}^T\tilde{F}_C - \tilde{F}_C\tilde{B}_{\mathrm{Im}}\big) \geqslant 0. \tag{A8}$$

Since $F_C \leqslant F_Q$, we have $\tilde{F}_C \leqslant I$, thus $\tilde{F}_C \geqslant \tilde{F}_C^2$ and $\tilde{B}_{\mathrm{Im}}^T \tilde{F}_C \tilde{B}_{\mathrm{Im}} \geqslant \tilde{B}_{\mathrm{Im}}^T \tilde{F}_C^2 \tilde{B}_{\mathrm{Im}}$. We then have

$$I - \tilde{F}_C - \tilde{B}_{\mathrm{Im}}^T \tilde{F}_C^2 \tilde{B}_{\mathrm{Im}} + i\big(\tilde{F}_{\mathrm{Im}} + \tilde{B}_{\mathrm{Im}}^T \tilde{F}_C - \tilde{F}_C \tilde{B}_{\mathrm{Im}}\big) \geqslant 0. \quad \text{(A9)}$$

Now denote $\tilde{F}_C \tilde{B}_{\mathrm{Im}}$ as D:

$$I - \tilde{F}_C - D^T D + i(\tilde{F}_{\mathrm{Im}} + D^T - D) \geqslant 0. \quad \text{(A10)}$$

Since any 2×2 principal submatrix of a positive-semidefinite matrix is also positive semidefinite,

$$\begin{pmatrix} 1 - (\tilde{F}_C)_{jj} - (D^T D)_{jj} & -(\tilde{F}_C)_{jk} - (D^T D)_{jk} \\ -(\tilde{F}_C)_{kj} - (D^T D)_{kj} & 1 - (\tilde{F}_C)_{kk} - (D^T D)_{kk} \end{pmatrix} + i\begin{pmatrix} 0 & (\tilde{F}_{\mathrm{Im}})_{jk} + D_{kj} - D_{jk} \\ -(\tilde{F}_{\mathrm{Im}})_{jk} - D_{kj} + D_{jk} & 0 \end{pmatrix} \quad \text{(A11)}$$

is then positive semidefinite. Note that $\tilde{F}_C$ and $D^T D$ are symmetric and the determination of a positive-semidefinite matrix is non-negative, we thus have

$$\begin{aligned} &[1 - (\tilde{F}_C)_{jj} - (D^T D)_{jj}][1 - (\tilde{F}_C)_{kk} - (D^T D)_{kk}] \\ &\quad \geqslant [(\tilde{F}_C)_{jk} + (D^T D)_{jk}]^2 + [(\tilde{F}_{\mathrm{Im}})_{jk} + D_{kj} - D_{jk}]^2, \end{aligned} \quad \text{(A12)}$$

from which we can get

$$\begin{aligned} &[1 - (\tilde{F}_C)_{jj} - (D^T D)_{jj}] + [1 - (\tilde{F}_C)_{kk} - (D^T D)_{kk}] \\ &\quad \geqslant 2\sqrt{[1 - (\tilde{F}_C)_{jj} - (D^T D)_{jj}][1 - (\tilde{F}_C)_{kk} - (D^T D)_{kk}]} \\ &\quad \geqslant 2\sqrt{[(\tilde{F}_C)_{jk} + (D^T D)_{jk}]^2 + [(\tilde{F}_{\mathrm{Im}})_{jk} + D_{kj} - D_{jk}]^2} \\ &\quad \geqslant 2|(\tilde{F}_{\mathrm{Im}})_{jk} + D_{kj} - D_{jk}|, \end{aligned} \quad \text{(A13)}$$

i.e.,

$$\begin{aligned} &1 - (\tilde{F}_C)_{jj} + 1 - (\tilde{F}_C)_{kk} \\ &\quad \geqslant 2|(\tilde{F}_{\mathrm{Im}})_{jk} + D_{kj} - D_{jk}| + (D^T D)_{jj} + (D^T D)_{kk}. \end{aligned} \quad \text{(A14)}$$

As $(D^T D)_{jj} = \sum_p D_{pj}^2 \geqslant D_{kj}^2$ and $(D^T D)_{kk} = \sum_p D_{pk}^2 \geqslant D_{jk}^2$, we have

$$\begin{aligned} &(D^T D)_{jj} + (D^T D)_{kk} \\ &\quad = \sum_p \big(D_{pj}^2 + D_{pk}^2\big) \geqslant D_{kj}^2 + D_{jk}^2 \\ &\quad = \frac{1}{2}(D_{kj} - D_{jk})^2 + \frac{1}{2}(D_{kj} + D_{jk})^2, \end{aligned} \quad \text{(A15)}$$

and from $I + i\tilde{F}_{\mathrm{Im}} = F_Q^{-\frac{1}{2}} F F_Q^{-\frac{1}{2}} \geqslant 0$, we have $|(\tilde{F}_{\mathrm{Im}})_{jk}| \leqslant 1$. Thus,

$$\begin{aligned} &1 - (\tilde{F}_C)_{jj} + 1 - (\tilde{F}_C)_{kk} \\ &\quad \geqslant 2|(\tilde{F}_{\mathrm{Im}})_{jk} + D_{kj} - D_{jk}| + (D^T D)_{jj} + (D^T D)_{kk} \\ &\quad \geqslant 2|(\tilde{F}_{\mathrm{Im}})_{jk} + D_{kj} - D_{jk}| + \tfrac{1}{2}(D_{kj} - D_{jk})^2 \\ &\quad \geqslant \tfrac{1}{2}|(\tilde{F}_{\mathrm{Im}})_{jk}|^2, \end{aligned} \quad \text{(A16)}$$

where the last inequality we used the fact that $2|y + x| + \frac{1}{2}x^2 \geqslant \frac{1}{2}y^2$ when $|y| \leqslant 1$ since

$$\begin{aligned} 2|y + x| + \tfrac{1}{2}x^2 &= 2|y + x| + \tfrac{1}{2}(y + x - y)^2 \\ &= 2|y + x| + \tfrac{1}{2}(y + x)^2 - y(x + y) + \tfrac{1}{2}y^2 \\ &\geqslant 2|y + x| - |y(x + y)| + \tfrac{1}{2}y^2 \\ &= (2 - |y|)|x + y| + \tfrac{1}{2}y^2 \\ &\geqslant \tfrac{1}{2}y^2. \end{aligned} \quad \text{(A17)}$$

This provides a tradeoff relation between $(\tilde{F}_C)_{jj}$ and $(\tilde{F}_C)_{kk}$. When $F_{\mathrm{Im}} = 0$, the quantum Cramér-Rao bound is saturable, F_C can reach F_Q, in this case $\tilde{F}_C = I$, $(\tilde{F}_C)_{jj}$ and $(\tilde{F}_C)_{kk}$ can reach the maximal value simultaneously, which is 1. When $(\tilde{F}_{\mathrm{Im}})_{jk} \neq 0$, $(\tilde{F}_C)_{jj}$ and $(\tilde{F}_C)_{kk}$ can not simultaneously reach 1, Eq. (A16) puts a tradeoff between them.

By summing Eq. (A16) over different choice of j, k directly, we can get

$$\begin{aligned} &2(n-1)\sum_j [1 - (\tilde{F}_C)_{jj}] \\ &\quad \geqslant \frac{1}{2}\sum_{j,k,j\neq k} |(\tilde{F}_{\mathrm{Im}})_{jk}|^2 = \frac{1}{2}\|\tilde{F}_{\mathrm{Im}}\|_F^2, \end{aligned} \quad \text{(A18)}$$

which gives

$$\mathrm{Tr}(\tilde{F}_C) \leqslant n - \frac{1}{4(n-1)}\|\tilde{F}_{\mathrm{Im}}\|_F^2, \quad \text{(A19)}$$

where $\|\tilde{F}_{\mathrm{Im}}\|_F^2 = \mathrm{Tr}(\tilde{F}_{\mathrm{Im}}^T \tilde{F}_{\mathrm{Im}})$. This can be rewritten as

$$\begin{aligned} \mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] &\leqslant n - \frac{1}{4(n-1)}\big\|F_Q^{-\frac{1}{2}} F_{\mathrm{Im}} F_Q^{-\frac{1}{2}}\big\|_F^2 \\ &= n - \frac{1}{4(n-1)}\mathrm{Tr}\big(F_Q^{-1} F_{\mathrm{Im}}^T F_Q^{-1} F_{\mathrm{Im}}\big). \end{aligned} \quad \text{(A20)}$$

The same relation can be obtained by including the number of copies of the state ν explicitly; essentially just replace F_Q and F_{Im} with νF_Q and νF_{Im}. The tradeoff relation with ν copies of the state is then

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] \leqslant n - \frac{1}{4(n-1)}\big\|F_Q^{-\frac{1}{2}} F_{\mathrm{Im}} F_Q^{-\frac{1}{2}}\big\|_F^2. \quad \text{(A21)}$$

When the number of the parameters $n \geqslant 3$, the tradeoff can be tightened by keeping all terms in $(D^T D)_{jj}$ and $(D^T D)_{kk}$ in Eq. (A14) as

$$\begin{aligned} &\sum_{j,k,j\neq k} [1 - (\tilde{F}_C)_{jj} + 1 - (\tilde{F}_C)_{kk}] \\ &\quad \geqslant \sum_{j,k,j\neq k} [2|(\tilde{F}_{\mathrm{Im}})_{jk} + D_{kj} - D_{jk}| \\ &\qquad + (D^T D)_{jj} + (D^T D)_{kk}], \end{aligned} \quad \text{(A22)}$$

where

$$(D^T D)_{jj} + (D^T D)_{kk} = \sum_p \big(D_{pj}^2 + D_{pk}^2\big), \quad \text{(A23)}$$

which not only includes the correlations between the j, kth entry, but also with other entries. By summing over all choice of j, k, we have

$$\begin{aligned}2(n-1)\sum_j[1-(\tilde{F}_C)_{jj}] &\geqslant \sum_{j,k,j\neq k} 2|(\tilde{F}_{\rm Im})_{jk}+D_{kj}-D_{jk}|+2(n-1)\sum_j (D^TD)_{jj}\\ &= \sum_{j,k,j\neq k} 2|(\tilde{F}_{\rm Im})_{jk}+D_{kj}-D_{jk}|+2(n-1)\sum_{j,k} D_{jk}^2\\ &\geqslant \sum_{j,k,j\neq k}\left\{2|(\tilde{F}_{\rm Im})_{jk}+D_{kj}-D_{jk}|+(n-1)\left(D_{jk}^2+D_{kj}^2\right)\right\}\\ &= \sum_{j,k,j\neq k}\left\{2|(\tilde{F}_{\rm Im})_{jk}+D_{kj}-D_{jk}|+\frac{n-1}{2}(D_{kj}-D_{jk})^2+\frac{n-1}{2}(D_{kj}+D_{jk})^2\right\}\\ &\geqslant \frac{2(n-2)}{n-1}\sum_{j,k,j\neq k}|(\tilde{F}_{\rm Im})_{jk}|^2\\ &= \frac{2(n-2)}{n-1}\|\tilde{F}_{\rm Im}\|_F^2, \end{aligned} \tag{A24}$$

where in the last inequality we used the fact that

$$\begin{aligned}2|y+x|+\frac{n-1}{2}x^2 &= 2|y+x|+\frac{n-1}{2}(y+x-y)^2\\ &= 2|y+x|+\frac{n-1}{2}(y+x)^2-(n-1)y(x+y)+\frac{n-1}{2}y^2\\ &\geqslant \frac{n-1}{2}(y+x)^2+2|y||y+x|-(n-1)|y(x+y)|+\frac{n-1}{2}y^2\\ &= \frac{n-1}{2}(y+x)^2-(n-3)|y||x+y|+\frac{n-1}{2}y^2\\ &= \frac{n-1}{2}\left(|y+x|-\frac{n-3}{n-1}|y|\right)^2+\frac{2(n-2)}{n-1}y^2\\ &\geqslant \frac{2(n-2)}{n-1}y^2. \end{aligned} \tag{A25}$$

This then gives a tradeoff relation on $\tilde{F}_C$ as

$$\mathrm{Tr}(\tilde{F}_C)\leqslant n-\frac{n-2}{(n-1)^2}\|\tilde{F}_{\rm Im}\|_F^2. \tag{A26}$$

With ν copies of the state, this can be equivalently written

$$\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\right]\leqslant n-\frac{n-2}{(n-1)^2}\left\|F_Q^{-\frac12}F_{\rm Im}F_Q^{-\frac12}\right\|_F^2. \tag{A27}$$

The bound can be further improved. From Eq. (A10),

$$I-\tilde{F}_C-D^TD+i(\tilde{F}_{\rm Im}+D^T-D)\geqslant 0, \tag{A28}$$

we have

$$I-\tilde{F}_C-D^TD\geqslant -i(\tilde{F}_{\rm Im}+D^T-D), \tag{A29}$$

from which we can obtain

$$\mathrm{Tr}(I-\tilde{F}_C-D^TD)\geqslant \|\tilde{F}_{\rm Im}+D^T-D\|_1. \tag{A30}$$

Note that $\tilde{F}_{\rm Im}+D^T-D$ is skew symmetric with purely imaginary eigenvalues, and the singular values are just the amplitude of the eigenvalues as $\{|\lambda_1|,\ldots,|\lambda_n|\}$. Since

$$-i(\tilde{F}_{\rm Im}+D^T-D)\leqslant I-\tilde{F}_C-D^TD\leqslant I, \tag{A31}$$

we have $|\lambda_j|\leqslant 1$, thus $|\lambda_j|\geqslant|\lambda_j|^2$. As

$$\begin{aligned}\|\tilde{F}_{\rm Im}+D^T-D\|_1 &= \sum_{j=1}^n|\lambda_j|\geqslant\sum_{j=1}^n|\lambda_j|^2\\ &= \mathrm{Tr}[(\tilde{F}_{\rm Im}+D^T-D)^T(\tilde{F}_{\rm Im}+D^T-D)]\\ &= \sum_{jk}[(\tilde{F}_{\rm Im}+D^T-D)_{jk}]^2, \end{aligned} \tag{A32}$$

from Eq. (A30) we then have

$$\begin{aligned}\mathrm{Tr}(I-\tilde{F}_C) &\geqslant \mathrm{Tr}(D^TD)+\|\tilde{F}_{\rm Im}+D^T-D\|_1\\ &\geqslant \sum_k(D^TD)_{kk}+\sum_{jk}[(\tilde{F}_{\rm Im}+D^T-D)_{jk}]^2\\ &= \sum_{jk}D_{jk}^2+[(\tilde{F}_{\rm Im})_{jk}+D_{kj}-D_{jk}]^2\\ &= \sum_{jk}\frac12\left(D_{jk}^2+D_{kj}^2\right)+[(\tilde{F}_{\rm Im})_{jk}+D_{kj}-D_{jk}]^2\\ &\geqslant \sum_{jk}\frac14(D_{jk}-D_{kj})^2+[(\tilde{F}_{\rm Im})_{jk}+D_{kj}-D_{jk}]^2 \end{aligned}$$

$$\geqslant \sum_{jk} \frac{1}{5}(\tilde{F}_{\rm Im})^2_{jk} = \frac{1}{5}\|\tilde{F}_{\rm Im}\|_F^2, \tag{A33}$$

where in the last inequality we used the fact that

$$\begin{aligned}\tfrac{1}{4}x^2 + (y+x)^2 &= \tfrac{5}{4}x^2 + 2xy + y^2 \\ &= \tfrac{5}{4}\big(x + \tfrac{4}{5}y\big)^2 + \tfrac{1}{5}y^2 \\ &\geqslant \tfrac{1}{5}y^2.\end{aligned} \tag{A34}$$

We thus have

$$\mathrm{Tr}(\tilde{F}_C) \leqslant n - \tfrac{1}{5}\|\tilde{F}_{\rm Im}\|_F^2. \tag{A35}$$

For ν copies of the state, this gives the tradeoff relation

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big] \leqslant n - \frac{1}{5}\big\|F_Q^{-\frac{1}{2}}F_{\rm Im}F_Q^{-\frac{1}{2}}\big\|_F^2, \tag{A36}$$

which is tighter than Eq. (A26) when $n \geqslant 5$.

APPENDIX B: CONNECTION BETWEEN THE PARTIAL COMMUTATIVE CONDITION AND THE WEAK COMMUTATIVE CONDITION

Here we show

$$\lim_{p\to\infty} \frac{\big\|\sqrt{\rho_x^{\otimes p}}[L_{jp}, L_{kp}]\sqrt{\rho_x^{\otimes p}}\big\|_1}{p} = |\mathrm{Tr}(\rho_x[L_j, L_k])|. \tag{B1}$$

We write the state in the eigenvalue decomposition as $\rho_x = \sum_{q=1}^m \lambda_q|\Psi_q\rangle\langle\Psi_q|$ with $\lambda_q > 0$ and $\sum_{q=1}^m \lambda_q = 1$. Then $\sqrt{\rho_x} = \sum_q \sqrt{\lambda_q}|\Psi_q\rangle\langle\Psi_q|$,

$$\begin{aligned}&\big\|\sqrt{\rho_x^{\otimes p}}[L_{jp}, L_{kp}]\sqrt{\rho_x^{\otimes p}}\big\|_1 \\ &= \left\|\sqrt{\rho_x^{\otimes p}}\sum_{r=1}^p \big[L_j^{(r)}, L_k^{(r)}\big]\sqrt{\rho_x^{\otimes p}}\right\|_1 \\ &= \left\|\sum_{r=1}^p \rho_x^{\otimes(r-1)} \otimes (\sqrt{\rho_x}[L_j, L_k]\sqrt{\rho_x}) \otimes \rho_x^{\otimes(p-r)}\right\|_1,\end{aligned} \tag{B2}$$

where $L_j^{(r)} = I^{\otimes(r-1)} \otimes L_j \otimes I^{\otimes(p-r)}$. The support of $\sum_{r=1}^p \rho_x^{\otimes(r-1)} \otimes (\sqrt{\rho_x}[L_j, L_k]\sqrt{\rho_x}) \otimes \rho_x^{\otimes(p-r)}$ is in the subspace spanned by $\{|\Psi_{v_1}\Psi_{v_2}\ldots\Psi_{v_p}\rangle\}$, with $|\Psi_{v_1}\rangle, \ldots, |\Psi_{v_p}\rangle \in \{|\Psi_1\rangle, \ldots, |\Psi_m\rangle\}$, where $\{|\Psi_1\rangle, \ldots, |\Psi_m\rangle\}$ are the eigenvectors of ρ_x with nonzero eigenvalues. We can focus on the support space and calculate the entries of $\sum_{r=1}^p \rho_x^{\otimes(r-1)} \otimes \sqrt{\rho_x}[L_j, L_k]\sqrt{\rho_x} \otimes \rho_x^{\otimes(p-r)}$ in the basis of $|\Psi_{v_1}\Psi_{v_2}\ldots\Psi_{v_p}\rangle$ with $v_1, \ldots, v_p \in \{1, \ldots, m\}$ and show that when $p \to \infty$, $\frac{\|\sqrt{\rho_x^{\otimes p}}[L_{jp}, L_{kp}]\|\sqrt{\rho_x^{\otimes p}}\|_1}{p} \to |\mathrm{Tr}(\rho_x[L_j, L_k])|$.

The entries of $\sqrt{\rho_x^{\otimes p}}[L_{jp}, L_{kp}]|\sqrt{\rho_x^{\otimes p}}$ are given by

$$\begin{aligned}&\langle\Psi_{\tilde{v}_1}\ldots\Psi_{\tilde{v}_p}|\sum_{r=1}^p \rho_x^{\otimes(r-1)} \\ &\otimes\sqrt{\rho_x}[L_j, L_k]\sqrt{\rho_x} \otimes \rho_x^{\otimes(p-r)}\big|\Psi_{v_1}\ldots\Psi_{v_p}\big\rangle \\ &= \sum_{r=1}^p \left[\big\langle\Psi_{\tilde{v}_r}\big|\sqrt{\rho_x}[L_j, L_k]\sqrt{\rho_x}\big|\Psi_{v_r}\big\rangle \prod_{q\neq r}\big(\delta_{\tilde{v}_q}^{v_q}\lambda_{v_q}\big)\right].\end{aligned} \tag{B3}$$

It is easy to see that when the indices $\{v_1, v_2\ldots, v_p\}$ and $\{\tilde{v}_1, \ldots, \tilde{v}_p\}$ differ at two or more entries, the corresponding matrix entry equals to 0. When the two indices differ at only one entry, for example, $v_r \neq \tilde{v}_r$ but $v_q = \tilde{v}_q$ for all $q \neq r$, the corresponding matrix entry equals to

$$\begin{aligned}&\big\langle\Psi_{\tilde{v}_r}\big|\sqrt{\rho_x}[L_j, L_k]\sqrt{\rho_x}\big|\Psi_{v_r}\big\rangle \prod_{q\neq r}\lambda_{\tilde{v}_q} \\ &= \frac{\big\langle\Psi_{\tilde{v}_r}\big|\sqrt{\rho_x}[L_j, L_k]\sqrt{\rho_x}\big|\Psi_{v_r}\big\rangle}{\lambda_{\tilde{v}_r}} \prod_{q=1}^p \lambda_{\tilde{v}_q}.\end{aligned} \tag{B4}$$

When the indices $\{v_1, v_2\ldots, v_p\}$ and $\{\tilde{v}_1, \ldots, \tilde{v}_p\}$ are the same, we get the diagonal entries of $\sqrt{\rho_x^{\otimes p}}[L_{jp}, L_{kp}]|\sqrt{\rho_x^{\otimes p}}$ as

$$\begin{aligned}&\sum_{r=1}^p \left(\big\langle\Psi_{v_r}\big|\sqrt{\rho_x}[L_j, L_k]\sqrt{\rho_x}\big|\Psi_{v_r}\big\rangle \prod_{q\neq r}\lambda_{v_q}\right) \\ &= \sum_{r=1}^p \left(\big\langle\Psi_{v_r}\big|[L_j, L_k]\big|\Psi_{v_r}\big\rangle \prod_{q=1}^p \lambda_{v_q}\right) \\ &= \left(\prod_{q=1}^p \lambda_{v_q}\right)\sum_{r=1}^p \big\langle\Psi_{v_r}\big|[L_j, L_k]\big|\Psi_{v_r}\big\rangle.\end{aligned} \tag{B5}$$

Next, we write $\sqrt{\rho_x^{\otimes p}}[L_{jp}, L_{kp}]|\sqrt{\rho_x^{\otimes p}} = D_p^{(jk)} + O_p^{(jk)}$ with $D_p^{(jk)}$ as the diagonal part of the matrix and $O_p^{(jk)}$ as the off-diagonal part of the matrix. We then use the inequality

$$\big\|D_p^{(jk)}\big\|_1 \leqslant \big\|D_p^{(jk)} + O_p^{(jk)}\big\|_1 \leqslant \big\|D_p^{(jk)}\big\|_1 + \big\|O_p^{(jk)}\big\|_1 \tag{B6}$$

to bound $\|\sqrt{\rho_x^{\otimes p}}[L_{jp}, L_{kp}]|\sqrt{\rho_x^{\otimes p}}\|_1$, where the first inequality comes from the fact that for any matrix M, $\|M\|_1 \geqslant \sum_q |M_{qq}|$, and for diagonal matrix $\|D_p^{(jk)}\|_1 = \sum_q |(D_p^{(jk)})_{qq}|$, the second inequality is from the triangle inequality of the trace norm.

The singular values of the diagonal matrix, $D_p^{(jk)}$, are just the absolute value of the diagonal entries, which are $\{(\prod_{r=1}^p \lambda_{v_r})|\sum_{r=1}^p\langle\Psi_{v_r}|[L_j, L_k]|\Psi_{v_r}\rangle|\}$. These entries can be interpreted as the absolute value of the summation of p randomly chosen $\langle\Psi_{v_r}|[L_j, L_k]|\Psi_{v_r}\rangle$ multiplied with the corresponding probabilities, where each term $\langle\Psi_{v_r}|[L_j, L_k]|\Psi_{v_r}\rangle$ is selected with probability λ_{v_r}. For a given diagonal entry with a particular choice of p terms, $|\sum_{r=1}^p\langle\Psi_{v_r}|[L_j, L_k]|\Psi_{v_r}\rangle|$, the associated probability is $\prod_{r=1}^p \lambda_{v_r}$. $\|D_p^{(jk)}\|_1$, which equals to the summation of the absolute value of all diagonal entries, then corresponds to the expected value of $|\sum_{r=1}^p\langle\Psi_{v_r}|[L_j, L_k]|\Psi_{v_r}\rangle|$ with each $|\Psi_{v_r}\rangle$ selected with probability λ_{v_r}. When $p \to \infty$, by the law of large numbers, $\frac{\sum_{r=1}^p\langle\Psi_{v_r}|[L_j, L_k]|\Psi_{v_r}\rangle}{p}$ converges to the expected value of

$\langle\Psi_{v_r}|[L_j,L_k]|\Psi_{v_r}\rangle$, i.e., with probability one

$$\begin{aligned}\frac{\sum_{r=1}^{p}\langle\Psi_{v_r}|[L_j,L_k]|\Psi_{v_r}\rangle}{p} &\to E\big[\langle\Psi_{v_r}|[L_j,L_k]|\Psi_{v_r}\rangle\big]\\ &=\sum_{q=1}^{m}\lambda_q\langle\Psi_q|[L_j,L_k]|\Psi_q\rangle\\ &=\sum_{q=1}^{m}\lambda_q\mathrm{Tr}(|\Psi_q\rangle\langle\Psi_q|[L_j,L_k])\\ &=\mathrm{Tr}(\rho_x[L_j,L_k]).\end{aligned}\tag{B7}$$

Thus, when $p\to\infty$,

$$\begin{aligned}\|D_p^{(jk)}\|_1 &= E\left[\left|\sum_{r=1}^{p}\langle\Psi_{v_r}|[L_j,L_k]|\Psi_{v_r}\rangle\right|\right]\\ &\to 1\times\left|E\left[\sum_{r=1}^{p}\langle\Psi_{v_r}|[L_j,L_k]|\Psi_{v_r}\rangle\right]\right|\\ &= p|\mathrm{Tr}(\rho_x[L_j,L_k])|.\end{aligned}\tag{B8}$$

For the off-diagonal part, note that for any matrix, we have $\|M\|_1\leqslant\sum_j\sqrt{\sum_k|M_{jk}|^2}$, and

$$\left|\frac{\langle\Psi_{\tilde v_r}|\sqrt{\rho_x}[L_j,L_k]\sqrt{\rho_x}|\Psi_{v_r}\rangle}{\lambda_{\tilde v_r}}\prod_{q=1}^{p}\lambda_{\tilde v_q}\right|\leqslant l_{\max}\textstyle\prod_{q=1}^{p}\lambda_{\tilde v_q},\tag{B9}$$

where $l_{\max}=\max_{\tilde v_r\neq v_r}\{|\frac{\langle\Psi_{\tilde v_r}|\sqrt{\rho_x}[L_j,L_k]\sqrt{\rho_x}|\Psi_{v_r}\rangle}{\lambda_{\tilde v_r}}|\}$, we then have

$$\begin{aligned}\|O_p^{(jk)}\|_1 &\leqslant\sum_{\tilde v_1,\dots,\tilde v_p}\sqrt{\sum_{r=1}^{p}\sum_{v_r\neq\tilde v_r}l_{\max}^2\prod_{q=1}^{p}\lambda_{\tilde v_q}^2}\\ &=\sum_{\tilde v_1,\dots,\tilde v_p}\sqrt{p(m-1)l_{\max}^2\prod_{q=1}^{p}\lambda_{\tilde v_q}^2}\\ &=\sqrt{(m-1)p}\,l_{\max}\sum_{\tilde v_1,\dots,\tilde v_p}\prod_{q=1}^{p}\lambda_{\tilde v_q}\\ &=\sqrt{(m-1)p}\,l_{\max}\prod_{q=1}^{p}\left(\sum_{\tilde v_q=1}^{m}\lambda_{\tilde v_q}\right)\\ &=\sqrt{(m-1)p}\,l_{\max}.\end{aligned}\tag{B10}$$

Thus, when $p\to\infty$,

$$\begin{aligned}\frac{\|D_p^{(jk)}+O_p^{(jk)}\|_1}{p}&\geqslant\frac{\|D_p^{(jk)}\|_1}{p}=|\mathrm{Tr}(\rho_x[L_j,L_k])|,\\ \frac{\|D_p^{(jk)}+O_p^{(jk)}\|_1}{p}&\leqslant\frac{\|D_p^{(jk)}\|_1+\|O_p^{(jk)}\|_1}{p}\leqslant|\mathrm{Tr}(\rho_x[L_j,L_k])|\\ &\quad+\frac{\sqrt{(m-1)}l_{\max}}{\sqrt{p}},\end{aligned}\tag{B11}$$

i.e.,

$$\begin{aligned}|\mathrm{Tr}(\rho_x[L_j,L_k])|&\leqslant\frac{\|\sqrt{\rho_x^{\otimes p}}[L_{jp},L_{kp}]\sqrt{\rho_x^{\otimes p}}\|_1}{p}\\ &\leqslant|\mathrm{Tr}(\rho_x[L_j,L_k])|+\frac{\sqrt{(m-1)}l_{\max}}{\sqrt{p}}.\end{aligned}\tag{B12}$$

From which it is easy to see that $\lim_{p\to\infty}\frac{\|\sqrt{\rho_x^{\otimes p}}[L_{jp},L_{kp}]\sqrt{\rho_x^{\otimes p}}\|_1}{p}=|\mathrm{Tr}(\rho_x[L_j,L_k])|$. The condition $\frac{\|\sqrt{\rho_x^{\otimes p}}[L_{jp},L_{kp}]\sqrt{\rho_x^{\otimes p}}\|_1}{p}=0$, then reduces to the weak commutative condition $\mathrm{Tr}(\rho_x[L_j,L_k])=0$, when $p\to\infty$.

It can also be seen that $\frac{\|D_p^{(jk)}\|_1}{p}$ provides a lower bound on $\frac{\|\sqrt{\rho_x^{\otimes p}}[L_{jp},L_{kp}]\sqrt{\rho_x^{\otimes p}}\|_1}{p}$ and the difference between them is in the order of $O(\frac{1}{\sqrt{p}})$. We can thus use $\|D_p^{(jk)}\|_1$ to provide an alternative tradeoff relation, which is less tight but easier to compute. Under p-local measurements the tradeoff relation can be written as

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\big]\leqslant n-\frac{1}{4(n-1)}\left\|\frac{T_p}{p}\right\|_F^2,\tag{B13}$$

where

$$\begin{aligned}(T_p)_{jk}&=\frac{1}{2}\|D_p^{(jk)}\|_1\\ &=\frac{1}{2}\sum_{v_1,\dots,v_p}\left(\prod_{r=1}^{p}\lambda_{v_r}\right)\left|\sum_{r=1}^{p}\langle\Psi_{v_r}|[\tilde L_j,\tilde L_k]|\Psi_{v_r}\rangle\right|,\end{aligned}\tag{B14}$$

where $\tilde L_j=\sum_q(F_Q^{-\frac{1}{2}})_{jq}L_q$ and $\tilde L_k=\sum_q(F_Q^{-\frac{1}{2}})_{kq}L_q$. Compared to C_p, T_p is expressed only with operators on a single copy of the state. We note that T_p can be equivalently obtained by choosing the set of $\{|u_j\rangle\}$ in Eq. (49) as the eigenvectors of ρ_x, instead of the eigenvectors of $\sqrt{\rho_x}[\tilde L_j,\tilde L_k]\sqrt{\rho_x}$.

APPENDIX C: BOUND ON THE TRACE NORM

For completeness, here we include a proof for the inequality $\sum_j|M_{jj}|\leqslant\|M\|_1\leqslant\sum_j\sqrt{\sum_k|M_{jk}|^2}$, which is used in the derivation that $\frac{C_p}{p}=0$ reduces to the weak commutative condition when $p\to\infty$. We first show $\|M\|_1\leqslant\sum_j\sqrt{\sum_k|M_{jk}|^2}$. From the singular value decomposition $M=U\Lambda V$, we have

$$\|M\|_1=\mathrm{Tr}(\Lambda)=Tr(U^\dagger MV^\dagger)=\mathrm{Tr}(V^\dagger U^\dagger M)=\mathrm{Tr}(WM),\tag{C1}$$

where $W=V^\dagger U^\dagger$ is a unitary matrix. Note that

$$\begin{aligned}(WM)_{jj}&=\sqrt{|(WM)_{jj}|^2}\leqslant\sqrt{\sum_k|(WM)_{jk}|^2}\\ &=\|(WM)_j\|_2\\ &=\|WM_j\|_2\\ &=\|M_j\|_2\\ &=\sqrt{\sum_k|M_{jk}|^2},\end{aligned}\tag{C2}$$

where we used $(\dots)_j$ to denote the jth column of a matrix [thus $(WM)_j$ is the jth column of WM which equals to WM_j, W multiples the jth column of M], and $\|v\|_2=\sqrt{\sum_k|v_k|^2}$ as the l_2 norm for a vector. It is then straightforward to see

$$\|M\|_1=\mathrm{Tr}(WM)=\sum_j(WM)_{jj}\leqslant\sum_j\sqrt{\sum_k|M_{jk}|^2}.\tag{C3}$$

Next, we show $\sum_j |M_{jj}| \leqslant \|M\|_1$. From the singular value decomposition $M = U\Lambda V$, we have

$$M_{jj} = \sum_k U_{jk}\Lambda_{kk}V_{kj}, \tag{C4}$$

thus,

$$\begin{aligned}\sum_j |M_{jj}| &= \sum_j \left|\sum_k U_{jk}\Lambda_{kk}V_{kj}\right| \\ &\leqslant \sum_j\sum_k |U_{jk}\Lambda_{kk}V_{kj}| \\ &= \sum_k\sum_j \Lambda_{kk}|U_{jk}V_{kj}| \\ &\leqslant \sum_k \Lambda_{kk}\sqrt{\left(\sum_j |U_{jk}|^2\right)\left(\sum_j |V_{kj}|^2\right)} \\ &= \sum_k \Lambda_{kk} \\ &= \|M\|_1.\end{aligned} \tag{C5}$$

APPENDIX D: PROOF OF $\mathrm{Cov}_u \geqslant A_u$

For a mixed state ρ_x, with $x = (x_1, \ldots, x_n)$, given any POVM, $\{M_\alpha\}$, and any $|u\rangle$, we define Cov_u as a $n \times n$ matrix with the jkth entry given by

$$(\mathrm{Cov}_u)_{jk} = \sum_\alpha [\hat{x}_j(\alpha) - x_j][\hat{x}_k(\alpha) - x_k]\langle u|\sqrt{\rho_x}M_\alpha\sqrt{\rho_x}|u\rangle, \tag{D1}$$

and A_u as a $n \times n$ matrix with the jkth entry given by

$$\begin{aligned}(A_u)_{jk} &= \langle u|\sqrt{\rho_x}X_j^\dagger X_k\sqrt{\rho_x}|u\rangle \\ &= \frac{1}{2}\langle u|\sqrt{\rho_x}\{X_j, X_k\}\sqrt{\rho_x}|u\rangle \\ &\quad + i\frac{1}{2i}\langle u|\sqrt{\rho_x}[X_j, X_k]\sqrt{\rho_x}|u\rangle,\end{aligned} \tag{D2}$$

where $X_j = \sum_\alpha [\hat{x}_j(\alpha) - x_j]M_\alpha$ is locally unbiased.

We then have $\mathrm{Cov}_u \geqslant A_u$ since for any vector $b = (b_1, \ldots, b_n)^T$,

$$\begin{aligned}&b^\dagger \mathrm{Cov}_u b - b^\dagger A_u b \\ &= \langle u|\sum_{j,k} b_j^* b_k \left\{\sum_\alpha [\hat{x}_j(\alpha) - x_j][\hat{x}_k(\alpha) - x_k]\sqrt{\rho_x}M_\alpha\sqrt{\rho_x} - \sum_\beta [\hat{x}_j(\beta) - x_j]\sqrt{\rho_x}M_\beta \sum_\gamma [\hat{x}_k(\gamma) - x_k]M_\gamma]\sqrt{\rho_x}\right\}|u\rangle \\ &= \langle u|\sum_{j,k}\Bigg\{\sum_\alpha [\hat{x}_j(\alpha) - x_j]b_j^*[\hat{x}_k(\alpha) - x_k]b_k\sqrt{\rho_x}M_\alpha\sqrt{\rho_x} \\ &\quad - \sum_\beta [\hat{x}_j(\beta) - x_j]b_j^*\sqrt{\rho_x}M_\beta\left(\sum_\alpha M_\alpha\right)\sum_\gamma [\hat{x}_k(\gamma) - x_k]b_k M_\gamma]\sqrt{\rho_x}\Bigg\}|u\rangle \\ &= \langle u|\sum_\alpha \Bigg\{\left[\sum_j [\hat{x}_j(\alpha) - x_j]b_j^*\sqrt{\rho_x} - \sum_j\sum_\beta [\hat{x}_j(\beta) - x_j]b_j^*\sqrt{\rho_x}M_\beta\right]M_\alpha\Bigg[\sum_k [x_k(\alpha) - x_k]b_k\sqrt{\rho_x} \\ &\quad - \sum_k\sum_\gamma [\hat{x}_k(\gamma) - x_k]b_k M_\gamma\sqrt{\rho_x}\Bigg]\Bigg\}|u\rangle \\ &= \langle u|\sum_\alpha M^\dagger(b)M_\alpha M(b)]|u\rangle \geqslant 0,\end{aligned} \tag{D3}$$

where $M(b) = \sum_k [x_k(\alpha) - x_k]b_k\sqrt{\rho_x} - \sum_k\sum_\gamma [\hat{x}_k(\gamma) - x_k]b_k M_\gamma\sqrt{\rho_x}$.

APPENDIX E: TRADEOFF RELATIONS WITH RLDs

Let

$$S_u = \begin{pmatrix} A_u & B_u \\ B_u^\dagger & F_u\end{pmatrix} \geqslant 0, \tag{E1}$$

with $(A_u)_{jk} = \langle u|\sqrt{\rho_x}X_jX_k\sqrt{\rho_x}|u\rangle$, $(B_u)_{jk} = \langle u|\sqrt{\rho_x}X_jL_k^{R\dagger}\sqrt{\rho_x}|u\rangle$, $(F_u)_{jk} = \langle u|\sqrt{\rho_x}L_j^RL_k^{R\dagger}\sqrt{\rho_x}|u\rangle$, where L_j^R is the RLD corresponding to the parameter x_j.

If we choose a complete basis $\{|u_1\rangle, \ldots, |u_d\rangle\}$, and let $S = \sum_j S_{u_j} = \begin{pmatrix} A & B \\ B^\dagger & F^{\mathrm{RLD}}\end{pmatrix} \geqslant 0$, where $(A)_{jk} = \mathrm{Tr}(\rho_x X_j X_k)$, $(B)_{jk} = \mathrm{Tr}(\rho_x X_j L_k^{R\dagger}) = I$, $(F^{\mathrm{RLD}})_{jk} = \mathrm{Tr}(\rho_x L_j^R L_k^{R\dagger})$, we obtain the RLD bound

$$\mathrm{Cov}(\hat{x}) \geqslant A \geqslant (F^{\mathrm{RLD}})^{-1}. \tag{E2}$$

This can be equivalently written as

$$\mathrm{Cov}^{-1}(\hat{x}) \leqslant F^{\mathrm{RLD}} = F_{\mathrm{Re}}^{\mathrm{RLD}} + iF_{\mathrm{Im}}^{\mathrm{RLD}}, \tag{E3}$$

with $F_{\mathrm{Re}}^{\mathrm{RLD}}$ and $F_{\mathrm{Im}}^{\mathrm{RLD}}$ as the real and imaginary parts of F^{RLD}, respectively, $F_{\mathrm{Re}}^{\mathrm{RLD}} = \frac{1}{2}[F^{\mathrm{RLD}} + (F^{\mathrm{RLD}})^T]$ is real symmetric and $F_{\mathrm{Im}}^{\mathrm{RLD}} = \frac{1}{2}[F^{\mathrm{RLD}} - (F^{\mathrm{RLD}})^T]$ is real skew symmetric. By taking the transpose, we also have [note $\mathrm{Cov}(\hat{x})$ is symmetric]

$$\mathrm{Cov}^{-1}(\hat{x}) \leqslant (F^{\mathrm{RLD}})^T = F^{\mathrm{RLD}}_{\mathrm{Re}} - iF^{\mathrm{RLD}}_{\mathrm{Im}}, \tag{E4}$$

from which we get

$$F_Q^{-\frac{1}{2}}\mathrm{Cov}^{-1}(\hat{x})F_Q^{-\frac{1}{2}} \leqslant F_Q^{-\frac{1}{2}}F^{\mathrm{RLD}}_{\mathrm{Re}}F_Q^{-\frac{1}{2}} \pm iF_Q^{-\frac{1}{2}}F^{\mathrm{RLD}}_{\mathrm{Im}}F_Q^{-\frac{1}{2}}. \tag{E5}$$

Then for any vector $|w\rangle$, we have

$$\langle w|F_Q^{-\frac{1}{2}}\mathrm{Cov}^{-1}(\hat{x})F_Q^{-\frac{1}{2}}|w\rangle \leqslant \langle w|F_Q^{-\frac{1}{2}}F^{\mathrm{RLD}}_{\mathrm{Re}}F_Q^{-\frac{1}{2}}|w\rangle - \left|\langle w|F_Q^{-\frac{1}{2}}F^{\mathrm{RLD}}_{\mathrm{Im}}F_Q^{-\frac{1}{2}}|w\rangle\right|. \tag{E6}$$

By choosing $|w\rangle$ as all the eigenvectors of $F_Q^{-\frac{1}{2}}F^{\mathrm{RLD}}_{\mathrm{Im}}F_Q^{-\frac{1}{2}}$ and making a summation, we obtain the tradeoff relation from the standard RLD as

$$\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\right] \leqslant \mathrm{Tr}\left[F_Q^{-1}F^{\mathrm{RLD}}_{\mathrm{Re}}\right] - \left\|F_Q^{-\frac{1}{2}}F^{\mathrm{RLD}}_{\mathrm{Im}}F_Q^{-\frac{1}{2}}\right\|_1. \tag{E7}$$

When there are ν copies of the state, this gives

$$\frac{1}{\nu}\mathrm{Tr}\left[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})\right] \leqslant \mathrm{Tr}\left[F_Q^{-1}F^{\mathrm{RLD}}_{\mathrm{Re}}\right] - \left\|F_Q^{-\frac{1}{2}}F^{\mathrm{RLD}}_{\mathrm{Im}}F_Q^{-\frac{1}{2}}\right\|_1. \tag{E8}$$

The bound can be improved by taking transposes on any S_u. We choose a complete basis $\{|u_1\rangle, \ldots, |u_d\rangle\}$ as the orthonormal eigenvectors of $\sqrt{\rho_x}(L^R_jL^{R\dagger}_k - L^R_kL^{R\dagger}_j)\sqrt{\rho_x}$. As mentioned in the main text, for any $|u_q\rangle$, $\frac{1}{2i}\langle u_q|\sqrt{\rho_x}(L^R_jL^{R\dagger}_k - L^R_kL^{R\dagger}_j)\sqrt{\rho_x}|u_q\rangle$, which is the imaginary part of $(F_{u_q})_{jk}$ is a real number, which we denote as t^q_{jk}. We then define

$$\bar{S}_{u_q} := \begin{cases} S_{u_q} & \text{when} \quad t^q_{jk} \geqslant 0, \\ S^T_{u_q} & \text{when} \quad t^q_{jk} < 0. \end{cases} \tag{E9}$$

By summing $\bar{S}_{u_q}$ we get

$$\bar{S} = \sum_q \bar{S}_{u_q} = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{B}^\dagger & \bar{F}^{\mathrm{RLD}} \end{pmatrix}, \tag{E10}$$

where $\bar{B} = I + i\bar{B}_{\mathrm{Im}}$, $\bar{F}^{\mathrm{RLD}} = \sum_q \bar{F}_{u_q}$ with $\bar{F}_{u_q}$ equals to either F_{u_q} or $F^T_{u_q}$ so that the imaginary part of $(\bar{F}_{u_q})_{jk}$ is always positive. The imaginary part of the jkth entry of $\bar{F}^{\mathrm{RLD}}$ is then given by

$$(\bar{F}_{\mathrm{Im}})_{jk} = \tfrac{1}{2}\left\|\sqrt{\rho_x}\left(L^R_jL^{R\dagger}_k - L^R_kL^{R\dagger}_j\right)\sqrt{\rho_x}\right\|_1, \tag{E11}$$

and the real part of $\bar{F}^{\mathrm{RLD}}$ remains the same as $F^{\mathrm{RLD}}_{\mathrm{Re}}$.

By the Schur's complement we then have $\bar{F}^{\mathrm{RLD}} - \bar{B}^\dagger\mathrm{Cov}^{-1}(\hat{x})\bar{B} \geqslant 0$, which can be equivalently written as

$$\bar{F}^{\mathrm{RLD}}_{\mathrm{Re}} + i\bar{F}^{\mathrm{RLD}}_{\mathrm{Im}} - \left[\mathrm{Cov}^{-1}(\hat{x}) + \bar{B}^T_{\mathrm{Im}}\mathrm{Cov}^{-1}(\hat{x})\bar{B}_{\mathrm{Im}} + i\left(\bar{B}^T_{\mathrm{Im}}\mathrm{Cov}^{-1}(\hat{x}) - \mathrm{Cov}^{-1}(\hat{x})\bar{B}_{\mathrm{Im}}\right)\right] \geqslant 0. \tag{E12}$$

We first assume $F_Q = I$, in this case $\mathrm{Cov}^{-1}(\hat{x}) \leqslant F_Q = I$. Then by following the same procedure as previous, we denote $\mathrm{Cov}^{-1}(\hat{x})\bar{B}_{\mathrm{Im}}$ as D and get

$$\bar{F}^{\mathrm{RLD}}_{\mathrm{Re}} - \mathrm{Cov}^{-1}(\hat{x}) - D^TD + i\left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Im}} + D^T - D\right) \geqslant 0. \tag{E13}$$

By taking a 2×2 principal submatrix we have

$$\begin{pmatrix} \left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Re}}\right)_{jj} - \mathrm{Cov}^{-1}(\hat{x})_{jj} - (D^TD)_{jj} & -\mathrm{Cov}^{-1}(\hat{x})_{jk} - (D^TD)_{jk} \\ -\mathrm{Cov}^{-1}(\hat{x})_{kj} - (D^TD)_{kj} & \left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Re}}\right)_{kk} - \mathrm{Cov}^{-1}(\hat{x})_{kk} - (D^TD)_{kk} \end{pmatrix} + i\begin{pmatrix} 0 & \left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Im}}\right)_{jk} + D_{kj} - D_{jk} \\ -\left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Im}}\right)_{jk} - D_{kj} + D_{jk} & 0 \end{pmatrix} \geqslant 0. \tag{E14}$$

From the positiveness of the determinant, we have

$$\left[\left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Re}}\right)_{jj} - \mathrm{Cov}^{-1}(\hat{x})_{jj} - (D^TD)_{jj}\right]\left[\left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Re}}\right)_{kk} - \mathrm{Cov}^{-1}(\hat{x})_{kk} - (D^TD)_{kk}\right] \geqslant [\mathrm{Cov}^{-1}(\hat{x})_{jk} + (D^TD)_{jk}]^2 + \left[\left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Im}}\right)_{jk} + D_{kj} - D_{jk}\right]^2, \tag{E15}$$

from which we can get

$$\begin{aligned} &\left[\left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Re}}\right)_{jj} - \mathrm{Cov}^{-1}(\hat{x})_{jj} - (D^TD)_{jj}\right] + \left[\left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Re}}\right)_{kk} - \mathrm{Cov}^{-1}(\hat{x})_{kk} - (D^TD)_{kk}\right] \\ &\geqslant 2\sqrt{\left[\left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Re}}\right)_{jj} - \mathrm{Cov}^{-1}(\hat{x})_{jj} - (D^TD)_{jj}\right]\left[\left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Re}}\right)_{kk} - \mathrm{Cov}^{-1}(\hat{x})_{kk} - (D^TD)_{kk}\right]} \\ &\geqslant 2\sqrt{[\mathrm{Cov}^{-1}(\hat{x})_{jk} + (D^TD)_{jk}]^2 + \left[\left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Im}}\right)_{jk} + D_{kj} - D_{jk}\right]^2} \\ &\geqslant 2\left|\left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Im}}\right)_{jk} + D_{kj} - D_{jk}\right|, \end{aligned} \tag{E16}$$

i.e.,

$$\left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Re}}\right)_{jj} - \mathrm{Cov}^{-1}(\hat{x})_{jj} + \left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Re}}\right)_{kk} - \mathrm{Cov}^{-1}(\hat{x})_{kk} \geqslant 2\left|\left(\bar{F}^{\mathrm{RLD}}_{\mathrm{Im}}\right)_{jk} + D_{kj} - D_{jk}\right| + (D^TD)_{jj} + (D^TD)_{kk}. \tag{E17}$$

Again as $(D^TD)_{jj}=\sum_p D_{pj}^2\geqslant D_{kj}^2$ and $(D^TD)_{kk}=\sum_p D_{pk}^2\geqslant D_{jk}^2$, we have

$$\begin{aligned}(D^TD)_{jj}&+(D^TD)_{kk}\\ &=\sum_p\left(D_{pj}^2+D_{pk}^2\right)\geqslant D_{kj}^2+D_{jk}^2\\ &=\frac{1}{2}(D_{kj}-D_{jk})^2+\frac{1}{2}(D_{kj}+D_{jk})^2.\end{aligned}\tag{E18}$$

Thus,

$$\begin{aligned}&\left(\bar F_{\rm Re}^{\rm RLD}\right)_{jj}-{\rm Cov}^{-1}(\hat x)_{jj}+\left(\bar F_{\rm Re}^{\rm RLD}\right)_{kk}-{\rm Cov}^{-1}(\hat x)_{kk}\\ &\geqslant 2\left|\left(\bar F_{\rm Im}^{\rm RLD}\right)_{jk}+D_{kj}-D_{jk}\right|+(D^TD)_{jj}+(D^TD)_{kk}.\\ &\geqslant 2\left|\left(\bar F_{\rm Im}^{\rm RLD}\right)_{jk}+D_{kj}-D_{jk}\right|+\tfrac{1}{2}(D_{kj}-D_{jk})^2\\ &\geqslant\min\left\{\tfrac{1}{2}|(\bar F_{Im})_{jk}|^2,2\right\},\end{aligned}\tag{E19}$$

where in the last inequality we used the fact that when $|y|\leqslant 2$, $2|y+x|+\frac{1}{2}x^2\geqslant\frac{1}{2}y^2$ since

$$\begin{aligned}2|y&+x|+\tfrac{1}{2}x^2\\ &=2|y+x|+\tfrac{1}{2}(y+x-y)^2\\ &=2|y+x|+\tfrac{1}{2}(y+x)^2-y(x+y)+\tfrac{1}{2}y^2\\ &\geqslant 2|y+x|-|y(x+y)|+\tfrac{1}{2}y^2\\ &=(2-|y|)|x+y|+\tfrac{1}{2}y^2\\ &\geqslant\tfrac{1}{2}y^2,\end{aligned}\tag{E20}$$

while when $|y|\geqslant 2$, $2|y+x|+\frac{1}{2}x^2\geqslant 2$ since

$$\begin{aligned}2|y+x|+\tfrac{1}{2}x^2&\geqslant 2(|y|-|x|)+\tfrac{1}{2}x^2\\ &=\tfrac{1}{2}(|x|-2)^2-2+2|y|\\ &\geqslant 2|y|-2\\ &\geqslant 2.\end{aligned}\tag{E21}$$

From which we can get

$$\begin{aligned}&\left(\bar F_{\rm Re}^{\rm RLD}\right)_{jj}-{\rm Cov}^{-1}(\hat x)_{jj}+\left(\bar F_{\rm Re}^{\rm RLD}\right)_{kk}\\ &\quad-{\rm Cov}^{-1}(\hat x)_{kk}\geqslant\min\left\{\tfrac{1}{2}\left|\bar F_{\rm Im}^{\rm RLD}\right)_{jk}|^2,2\right\}.\end{aligned}\tag{E22}$$

By repeating the procedure for different choices of j,k and making a summation, we then get the tradeoff relation, under the parametrization that $F_Q=I$, as

$${\rm Tr}[{\rm Cov}^{-1}(\hat x)]\leqslant{\rm Tr}\left[\bar F_{\rm Re}^{\rm RLD}\right]-\frac{1}{4(n-1)}\left\|C_1^{\rm RLD}\right\|_F^2,\tag{E23}$$

with $(C_1^{\rm RLD})_{jk}=\min\{\frac{1}{2}\|\sqrt{\rho_x}(L_j^RL_k^{R\dagger}-L_k^RL_j^{R\dagger})\sqrt{\rho_x}\|_1,2\}$.

If we repeat the 1-local measurement on ν copies of the state, the tradeoff relation under the 1-local measurement, with the parametrization that $F_Q=I$, is then

$$\frac{1}{\nu}{\rm Tr}[{\rm Cov}^{-1}(\hat x)]\leqslant{\rm Tr}\left[F_{\rm Re}^{\rm RLD}\right]-\frac{1}{4(n-1)}\left\|C_1^{\rm RLD}\right\|_F^2.\tag{E24}$$

When $F_Q\neq I$ initially, we can first make a reparametrization with $\tilde x=F_Q^{-\frac{1}{2}}x$. The tradeoff relation in Eq. (E24) can then be expressed in the original parametrization as

$$\begin{aligned}&\frac{1}{\nu}{\rm Tr}\left[F_Q^{-1}{\rm Cov}(\hat x)^{-1}\right]\\ &\qquad\leqslant{\rm Tr}\left[F_Q^{-1}F_{\rm Re}^{\rm RLD}\right]-\frac{1}{4(n-1)}\left\|C_1^{\rm RLD}\right\|_F^2\end{aligned}\tag{E25}$$

with the entries of $C_1^{\rm RLD}$ given by

$$\left(C_1^{\rm RLD}\right)_{jk}=\min\left\{\tfrac{1}{2}\left\|\sqrt{\rho_x}\left(\tilde L_j^R\tilde L_k^{R\dagger}-\tilde L_k^R\tilde L_j^{R\dagger}\right)\sqrt{\rho_x}\right\|_1,2\right\},\tag{E26}$$

where $\tilde L_j^R=\sum_q(F_Q^{-\frac{1}{2}})_{jq}L_q^R$ and $\tilde L_k^R=\sum_q(F_Q^{-\frac{1}{2}})_{kq}L_q^R$.

For p-local measurements, we can similarly get

$$\frac{1}{\nu}{\rm Tr}\left[F_Q^{-1}{\rm Cov}^{-1}(\hat x)\right]\leqslant{\rm Tr}\left[F_Q^{-1}F_{\rm Re}^{\rm RLD}\right]-\frac{1}{4(n-1)}\left\|\frac{C_p^{\rm RLD}}{p}\right\|_F^2,\tag{E27}$$

where $(C_p^{\rm RLD})_{jk}=\min\{\frac{1}{2}\|\sqrt{\rho_x^{\otimes p}}(\tilde L_{jp}^R\tilde L_{kp}^{R\dagger}-\tilde L_{kp}^R\tilde L_{jp}^{R\dagger})\sqrt{\rho_x^{\otimes p}}\|_1,2p\}$.

APPENDIX F: EXAMPLE 2

Here we provided more detailed calculations for example 2. For mixed states $\rho_x=\frac{1}{3}I+\sum_j x_jG_j$ with $G_j=\frac{1}{2}\Lambda_j$, where $\{\Lambda_j\}_{j=1}^8$ are the Gell-Mann matrices,

$$\begin{aligned}\Lambda_1&=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad\Lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\\ \Lambda_3&=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\quad\Lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\\ \Lambda_5&=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},\quad\Lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\\ \Lambda_7&=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad\Lambda_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}.\end{aligned}\tag{F1}$$

If the parameters x_j are all close to 0, the SLDs and RLDs are all given by $L_j=3G_j$. Thus, the tradeoff relations from the SLDs and RLDs will be the same. The QFI matrix is given as $F_Q=F^{\rm RLD}=\frac{3}{2}I$, thus $\tilde L_j=\sqrt{\frac{2}{3}}L_j=\sqrt{6}G_j$. The entries of C_1 are given by

$$(C_1)_{jk}=\tfrac{1}{2}\|\sqrt{\rho_x}[\tilde L_j,\tilde L_k]\sqrt{\rho_x}\|_1=\|[G_j,G_k]\|_1,\tag{F2}$$

(a) definition:

	a_{ij}	
$a_{ij}-\lambda$	a_{ij}	$a_{ij}+\lambda$

(b) eigenvalues:

			0			
		$-\lambda$	0	λ		
	-2λ	$-\lambda$	0	λ	2λ	
-3λ	-2λ	$-\lambda$	0	λ	2λ	3λ

(c) multiplicity:

			1			
		1	1	1		
	1	2	3	2	1	
1	3	6	7	6	3	1

FIG. 5. Eigenvalues and multiplicities of $\sum_{r=1}^{p}[G_j^{(r)}, G_k^{(r)}]$.

from which the matrix form of C_1 can be computed as

$$C_1=\begin{pmatrix}0&1&1&\frac12&\frac12&\frac12&\frac12&0\\ 1&0&1&\frac12&\frac12&\frac12&\frac12&0\\ 1&1&0&\frac12&\frac12&\frac12&\frac12&0\\ \frac12&\frac12&\frac12&0&1&\frac12&\frac12&\frac{\sqrt3}{2}\\ \frac12&\frac12&\frac12&1&0&\frac12&\frac12&\frac{\sqrt3}{2}\\ \frac12&\frac12&\frac12&\frac12&\frac12&0&1&\frac{\sqrt3}{2}\\ \frac12&\frac12&\frac12&\frac12&\frac12&1&0&\frac{\sqrt3}{2}\\ 0&0&0&\frac{\sqrt3}{2}&\frac{\sqrt3}{2}&\frac{\sqrt3}{2}&\frac{\sqrt3}{2}&0\end{pmatrix}. \tag{F3}$$

Thus, we have

$$\begin{aligned}&\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big]\\ &\leqslant n-\frac{1}{4(n-1)}\|C_1\|_F^2\\ &=8-\frac{1}{28}\times 2\left(5\times 1+4\times\frac34+16\times\frac14\right)\\ &=\frac{50}{7}\approx 7.14.\end{aligned} \tag{F4}$$

For p-local measurements on ρ_x, the entries of C_p are given by

$$(C_p)_{jk}=\frac12\Big\|\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{jp},\tilde{L}_{kp}]\sqrt{\rho_x^{\otimes p}}\Big\|_1=\frac{1}{3^{p-1}}\|[G_{jp},G_{kp}]\|_1. \tag{F5}$$

For all j,k, the eigenvalues of $[G_j,G_k]$ are $\{-\lambda,0,\lambda\}$, where $\lambda=\frac12$ or $\frac14$ or $\frac{\sqrt3}{4}$. Suppose that the eigenvectors corresponding to eigenvalues $\{-\lambda,0,\lambda\}$ can be written as $\{|\Phi_l\rangle\}_{l\in\{-\lambda,0,\lambda\}}$. The eigenvectors of $[G_{jp},G_{kp}]=\sum_{r=1}^{p}[G_j^{(r)},G_k^{(r)}]$ are then given by $\otimes_{r=1}^{p}|\Phi_{l_r}\rangle$ with the corresponding eigenvalues $\sum_{r=1}^{p}l_r$, where $l_r\in\{-\lambda,0,\lambda\}$. The recursive relation to obtain the eigenvalues is depicted in Fig. 5(a), where in Fig. 5(b) a few possible values of $\sum_{r=1}^{p}l_r$ have been listed [note that the $(p+1)$th row in Fig. 5(b) corresponds to all possible values of $\sum_{r=1}^{p}l_r$]. The multiplicity of each eigenvalue can be obtained as Fig. 5(c), which is just the trinomial triangle that corresponds to the coefficients of $(1+x+x^2)^p$. Hence, the eigenvalues of $[G_{jp},G_{kp}]=\sum_{r=1}^{p}[G_j^{(r)},G_k^{(r)}]$ are λs with multiplicity $\binom{p}{s}_2$ for $s=-p,-p+1,\dots,p$, where $\binom{p}{s}_2=\sum_{i=0}^{p}(-1)^i\binom{p}{i}\binom{2p-2i}{p-s-i}$ is the trinomial coefficient.

Denote $\mathcal{N}_p=\sum_{s=0}^{p}s\binom{p}{s}_2$, we then have

$$(C_p)_{jk}=\frac{1}{3^{p-1}}\|[G_{jp},G_{kp}]\|_1=(C_1)_{jk}\frac{\mathcal{N}_p}{3^{p-1}}. \tag{F6}$$

The Frobenius norm of C_p is then given by

$$\begin{aligned}\|C_p\|_F&=\sqrt{\sum_{jk}(C_p)_{jk}^2}=\sqrt{\sum_{jk}\left((C_1)_{jk}\frac{1}{3^{p-1}}\mathcal{N}_p\right)^2}\\ &=\frac{1}{3^{p-1}}\mathcal{N}_p\sqrt{\sum_{jk}((C_1)_{jk})^2}=\frac{1}{3^{p-1}}\mathcal{N}_p\|C_1\|_F.\end{aligned} \tag{F7}$$

Using the tradeoff relations for p-local measurements, we have

$$\begin{aligned}&\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big]\\ &\leqslant n-\frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2\\ &=n-\frac{1}{4(n-1)}\|C_1\|_F^2\left(\frac{1}{p3^{p-1}}\mathcal{N}_p\right)^2\\ &=8-\frac67\left(\frac{1}{p3^{p-1}}\mathcal{N}_p\right)^2.\end{aligned} \tag{F8}$$

Specifically, for 2-local measurements,

$$\begin{aligned}&\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big]\\ &\leqslant n-\frac{1}{4(n-1)}\left\|\frac{C_2}{2}\right\|_F^2\\ &=8-\frac67\times\frac14\times\frac{16}{9}=\frac{160}{21}\approx 7.62,\end{aligned} \tag{F9}$$

and for 3-local measurements,

$$\begin{aligned}&\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big]\\ &\leqslant n-\frac{1}{4(n-1)}\left\|\frac{C_3}{3}\right\|_F^2\\ &=8-\frac67\times\frac19\times\frac{25}{9}=\frac{1462}{189}\approx 7.74.\end{aligned} \tag{F10}$$

If we choose the basis $\{|u\rangle_q\}$ as computational basis $|u\rangle_0 = |0\rangle$, $|u\rangle_1 = |1\rangle$, $|u\rangle_2 = |2\rangle$, the matrices F_{u_q} are given as

$$F_{u_0} = \frac{1}{2}\begin{pmatrix} 1 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} \\ 0 & 0 & 0 & 1 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix},$$

$$F_{u_1} = \frac{1}{2}\begin{pmatrix} 1 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 1 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \tag{F11}$$

$$F_{u_2} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{3} \end{pmatrix},$$

$$\bar{F}_{\mathrm{Im}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{F12}$$

Let $\bar{F} = F_{u_0} + F_{u_1}^T + F_{u_2}^T$, this gives a bound as

$$\begin{aligned} &\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \\ &\quad\leqslant n - \frac{(n-2)}{(n-1)^2}\|\bar{F}_{\mathrm{Im}}\|_F^2 \\ &\quad= 8 - \frac{6}{49}\times 4 \approx 7.51. \end{aligned} \tag{F13}$$

If we only estimate $\{x_1, x_2, x_4, x_5\}$, the associated matrices are given by the 4×4 submatrices of the original ones,

$$C_1 = \begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \end{pmatrix},$$

$$\bar{F}_{\mathrm{Im}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{F14}$$

which further gives $\|C_1\|_F = \sqrt{6}$, $\|\bar{F}_{\mathrm{Im}}\|_F = 2$. Then, we have

$$\begin{aligned} \frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] &\leqslant n - \frac{1}{4(n-1)}\|C_1\|_F^2 \\ &= 4 - \frac{1}{2} = \frac{7}{2} = 3.5, \end{aligned} \tag{F15}$$

$$\begin{aligned} \frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] &\leqslant n - \frac{(n-2)}{(n-1)^2}\|\bar{F}_{\mathrm{Im}}\|_F^2 \\ &= 4 - \frac{8}{9} = \frac{28}{9} \approx 3.11. \end{aligned} \tag{F16}$$

For p-local measurements, by following the same derivation as the previous case, we have

$$\begin{aligned} &\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \\ &\quad\leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2 \\ &\quad= n - \frac{1}{4(n-1)}\|C_1\|_F^2\left(\frac{1}{p3^{p-1}}\mathcal{N}_p\right)^2 \\ &\quad= 4 - \frac{1}{2}\left(\frac{1}{p3^{p-1}}\mathcal{N}_p\right)^2. \end{aligned} \tag{F17}$$

Specifically, for $p = 2$ we have

$$\begin{aligned} &\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \\ &\quad\leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_2}{2}\right\|_F^2 \\ &\quad= 4 - \frac{1}{2}\times\frac{1}{4}\times\frac{16}{9} = \frac{34}{9} \approx 3.78, \end{aligned} \tag{F18}$$

and for $p = 3$,

$$\begin{aligned} &\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \\ &\quad\leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_3}{3}\right\|_F^2 \\ &\quad= 4 - \frac{1}{2}\times\frac{1}{9}\times\frac{25}{9} = \frac{623}{162} \approx 3.85. \end{aligned} \tag{F19}$$

If we choose the basis $\{|u_q\rangle\}$ as the computational basis $|u_0\rangle = |00\rangle$, $|u_1\rangle = |01\rangle$, $|u_2\rangle = |02\rangle$, $|u_3\rangle = |10\rangle$, ..., $|u_8\rangle = |22\rangle$, the imaginary parts of the matrices F_{u_q} are given as

$$F_{u_0\mathrm{Im}} = \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & 0 \end{pmatrix},$$

$$F_{u_4\mathrm{Im}} = \begin{pmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F_{u_8\mathrm{Im}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 \end{pmatrix},$$

$$F_{u_1\mathrm{Im}} = F_{u_3\mathrm{Im}} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & 0 \end{pmatrix},$$

$$F_{u_2\mathrm{Im}} = F_{u_6\mathrm{Im}} = \frac{1}{2}\begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F_{u_5\mathrm{Im}} = F_{u_7\mathrm{Im}} = \frac{1}{2}\begin{pmatrix} 0 & -\frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 \end{pmatrix}. \tag{F20}$$

The optimal $\bar{F}_{\mathrm{Im}2}$ is then given by $\bar{F}_{\mathrm{Im}2} = F_{u_0\mathrm{Im}} + F_{u_4\mathrm{Im}}^T + F_{u_8\mathrm{Im}}^T + (F_{u_1\mathrm{Im}} + F_{u_3\mathrm{Im}}) + (F_{u_2\mathrm{Im}} + F_{u_6\mathrm{Im}}) + (F_{u_5\mathrm{Im}} + F_{u_7\mathrm{Im}})^T$, i.e.,

$$\bar{F}_{\mathrm{Im}2} = \begin{pmatrix} 0 & \frac{4}{3} & 0 & 0 \\ -\frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & -\frac{4}{3} & 0 \end{pmatrix}, \tag{F21}$$

which gives a tighter bound as

$$\begin{aligned} &\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \\ &\quad \leqslant n - \frac{(n-2)}{(n-1)^2}\left\|\frac{\bar{F}_{\mathrm{Im}2}}{2}\right\|_F^2 \\ &\quad = 4 - \frac{2}{9}\times\frac{1}{4}\times 4\times\frac{16}{9} = 4 - \frac{32}{81} \approx 3.60. \end{aligned} \tag{F22}$$

If we only estimate $\{x_1, x_2, x_5\}$, the associated matrices are given by the 3×3 submatrices of the original ones,

$$C_1 = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \quad \bar{F}_{\mathrm{Im}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{F23}$$

which further gives $\|C_1\|_F = \sqrt{3}$, $\|\bar{F}_{\mathrm{Im}}\|_F = \sqrt{2}$. Then, we have the tradeoff relations

$$\begin{aligned} \frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] &\leqslant n - \frac{1}{4(n-1)}\|C_1\|_F^2 \\ &= 3 - \frac{3}{8} = \frac{21}{8} = 2.625, \end{aligned} \tag{F24}$$

$$\begin{aligned} \frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] &\leqslant n - \frac{(n-2)}{(n-1)^2}\|\bar{F}_{\mathrm{Im}}\|_F^2 \\ &= 3 - \frac{1}{2} = \frac{5}{2} = 2.5. \end{aligned} \tag{F25}$$

For p-local measurements, we have

$$\begin{aligned} \frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] &\leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2 \\ &= n - \frac{1}{4(n-1)}\|C_1\|_F^2\left(\frac{1}{p3^{p-1}}\mathcal{N}_p\right)^2 \\ &= 3 - \frac{3}{8}\left(\frac{1}{p3^{p-1}}\mathcal{N}_p\right)^2. \end{aligned} \tag{F26}$$

Specifically, for $p = 2$,

$$\begin{aligned} &\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \\ &\quad \leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_2}{2}\right\|_F^2 \\ &\quad = 3 - \frac{3}{8}\times\frac{1}{4}\times\frac{16}{9} = \frac{17}{6} \approx 2.83, \end{aligned} \tag{F27}$$

and for $p = 3$,

$$\begin{aligned} &\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \\ &\quad \leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_3}{3}\right\|_F^2 \\ &\quad = 3 - \frac{3}{8}\times\frac{1}{9}\times\frac{25}{9} = \frac{623}{216} \approx 2.88. \end{aligned} \tag{F28}$$

For 2-local measurements, if we choose the basis $\{|u_q\rangle\}$ as the computational basis $|u_0\rangle = |00\rangle$, $|u_1\rangle = |01\rangle$, $|u_2\rangle = |02\rangle$, $|u_3\rangle = |10\rangle$, ..., $|u_8\rangle = |22\rangle$, the imaginary parts of the matrices F_{u_q} are given as

$$F_{u_0\mathrm{Im}} = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$F_{u_4\mathrm{Im}} = \begin{pmatrix} 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$F_{u_1\mathrm{Im}} = F_{u_3\mathrm{Im}} = F_{u_8\mathrm{Im}} = \mathbf{0},$$

$$F_{u_2\mathrm{Im}} = F_{u_6\mathrm{Im}} = \frac{1}{2}\begin{pmatrix} 0 & \frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$F_{u_5\mathrm{Im}} = F_{u_7\mathrm{Im}} = \frac{1}{2}\begin{pmatrix} 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{F29}$$

The optimal $\bar{F}_{\mathrm{Im}2}$ is then given by $\bar{F}_{\mathrm{Im}2} = F_{u_0\mathrm{Im}} + F_{u_4\mathrm{Im}}^T + (F_{u_2\mathrm{Im}} + F_{u_6\mathrm{Im}}) + (F_{u_5\mathrm{Im}} + F_{u_7\mathrm{Im}})^T$, i.e.,

$$\bar{F}_{\mathrm{Im}2} = \begin{pmatrix} 0 & \frac{4}{3} & 0 \\ -\frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{F30}$$

which gives

$$\begin{aligned} &\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \\ &\quad \leqslant n - \frac{(n-2)}{(n-1)^2}\left\|\frac{\bar{F}_{\mathrm{Im}2}}{2}\right\|_F^2 \\ &\quad = 3 - \frac{1}{4}\times\frac{1}{4}\times 2\times\frac{16}{9} = 3 - \frac{2}{9} \approx 2.78. \end{aligned} \tag{F31}$$

This is tighter than the bound given by C_2.

If we only estimate $\{x_1, x_2\}$, the associated matrices are given by the 2×2 submatrices of the original ones,

$$C_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{F32}$$

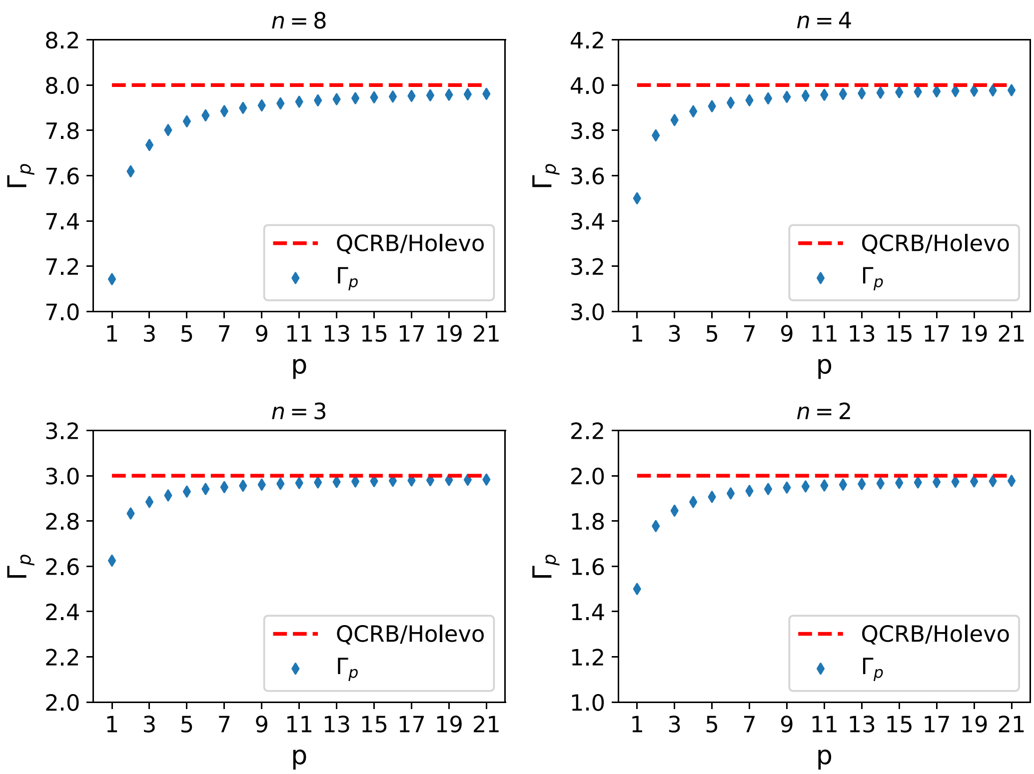

FIG. 6. Upper bound on Γ_p and the QCRB and Holevo bounds with the number of parameters equal to 8, 4, 3, 2, respectively.

which further gives $\|C_1\|_F = \sqrt{2}$. Then, we have the tradeoff relation

$$\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] \leqslant n - \frac{1}{4(n-1)}\|C_1\|_F^2 = 2 - \frac{1}{2} = \frac{3}{2}. \quad \text{(F33)}$$

For p-local measurements, we have

$$\begin{aligned}\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] &\leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2 \\ &= n - \frac{1}{4(n-1)}\|C_1\|_F^2\left(\frac{1}{p3^{p-1}}\mathcal{N}_p\right)^2 \\ &= 2 - \frac{1}{2}\left(\frac{1}{p3^{p-1}}\mathcal{N}_p\right)^2. \quad \text{(F34)}\end{aligned}$$

Specifically, for $p = 2$,

$$\begin{aligned}\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] &\leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_2}{2}\right\|_F^2 \\ &= 2 - \frac{1}{2}\times\frac{1}{4}\times\frac{16}{9} = \frac{16}{9} \approx 1.78, \quad \text{(F35)}\end{aligned}$$

and for $p = 3$,

$$\begin{aligned}\frac{1}{\nu}\mathrm{Tr}\big[F_Q^{-1}\mathrm{Cov}(\hat{x})^{-1}\big] &\leqslant n - \frac{1}{4(n-1)}\left\|\frac{C_3}{3}\right\|_F^2 \\ &= 2 - \frac{1}{2}\times\frac{1}{9}\times\frac{25}{9} = \frac{299}{162} \approx 1.85. \quad \text{(F36)}\end{aligned}$$

We plot the bound with different p in Fig. 6. It can be seen that the Holevo bound, which equals to the QCRB since the weak commutative condition holds in this case, is only achievable when $p \to \infty$.

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