Information Geometry under Hierarchical Quantum Measurement

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In most quantum technologies, measurements need to be performed on the parametrized quantum states to transform the quantum information to classical information. The measurements, however, inevitably distort the information. The characterization of the discrepancy is an important subject in quantum information science, which plays a key role in understanding the difference between the structures of quantum and classical informations. Here we analyze the difference in terms of the Fisher information metric and present a framework that can provide analytical bounds on the discrepancy under hierarchical quantum measurements. Specifically, we present a set of analytical bounds on the difference between the quantum and classical Fisher information metric under hierarchical p-local quantum measurements, which are measurements that can be performed collectively on at most p copies of quantum states. The results can be directly transformed to the precision limit in multiparameter quantum metrology, which leads to characterizations of the trade-off among the precision of different parameters. The framework also provides a coherent picture for various existing results by including them as special cases.

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Quantum measurement serves as the gateway between quantum information and classical information. In most quantum technologies, the information encoded in the parametrized quantum states needs to be extracted by the measurements. For example, in quantum metrology, the estimation of unknown parameters encoded in the quantum states is achieved through the measurements on the parametrized quantum states; in the variational quantum circuit certain information needs to be extracted via the measurements on the parametrized quantum states to update the circuit. Upon the measurements, however, certain properties of quantum information, such as noncommutativity, are lost. This inevitably induces distortions on the information structure. Understanding such distortion is an important subject in quantum information science which helps distinguish the structure of the quantum and classical information. It also helps to understand the maximal amount of information that can be extracted from quantum states.

In this Letter, we characterize the structure of the information in terms of the Fisher information metric [1–16] and study the distortion of the metric under hierarchical p-local quantum measurements, which are the measurements that can be performed collectively on at most p copies of quantum states. We note that the Fisher information metric is the only Riemannian metric in information geometry that is invariant under sufficient statistics [17], and has been employed in a broad range of applications [18], such as quantum metrology [1,2], quantum phase transition [19,20], entanglement witness [21,22], as well as the natural gradient and effective dimension in statistical learning [17,23].

We first introduce the quantum and classical Fisher information metric together with the existing results, then present an approach to characterize the achievable classical Fisher information metric under hierarchical p-local quantum measurements. This framework provides a systematical way to generate various analytical bounds on the difference between the quantum and classical Fisher information metric under general p-local measurements. The framework also includes various existing results, which were seemingly disconnected from each other previously, as special cases thus put them into a coherent picture.

Given d-dimensional parametrized quantum state ρ_x, with n parameters as $x=(x_1,\ldots,x_n)$, the jkth entry of the quantum Fisher information matrix (QFIM), denoted as $F_Q(x)$, is given by [1–3] $F_Q(x)_{jk}=\frac{1}{2}\mathrm{Tr}[\rho_x(L_jL_k+L_kL_j)]$, where $L_j(L_k)$ is the symmetric logarithmic derivative (SLD) with respect to the parameter $x_j(x_k)$ and can be obtained from the equation $(\partial\rho_x/\partial x_j)=\frac{1}{2}(\rho_xL_j+L_j\rho_x)$. The QFIM quantifies the quantum Fisher information metric, which is related to the Bures metric as [3] $D_B^2(\rho_x,\rho_{x+dx})=\frac{1}{4}dxF_Q(x)dx^T$, where $dx=(dx_1,\ldots,dx_n)$ are infinitesimal changes of the parameters and $D_B(\rho_1,\rho_2)=\sqrt{2-2\mathrm{Tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}}$ is the Bures distance [24]. The QFIM is additive with respect to the copies of the state; i.e., the QFIM of $\rho_x^{\otimes p}$, which we denote as F_{Qp}, equals to pF_Q.

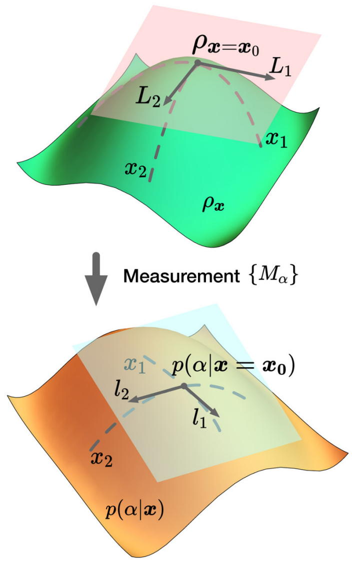

FIG. 1. The measurement transforms the parametrized quantum states to classical probability distribution where the classical Fisher information metric is always upper bounded by the quantum Fisher information metric.

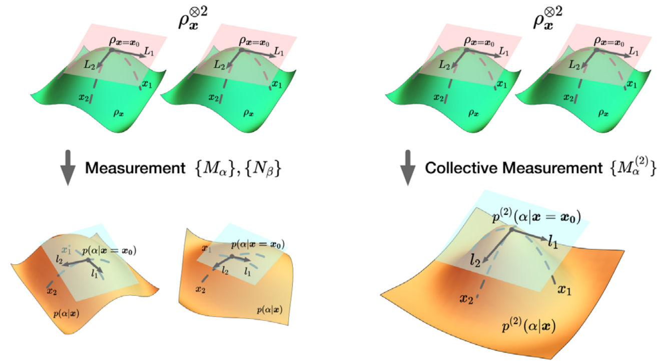

FIG. 2. The transformation of information geometry under 1-local and 2-local measurements, respectively. Since there is more freedom in the 2-local measurements, the classical Fisher information metric under the optimal 2-local measurement is in general less distorted than the metric under the 1-local measurements; i.e., the classical Fisher information matrix under 2-local measurements can be closer to the quantum Fisher information matrix. The classical Fisher information matrix under p-local measurements can get even closer when p increases.

General measurements that transform the parametrized quantum states to classical data are the positive operator-valued measurement(POVM), which is a set of $\{M_\alpha\}$ with $M_\alpha \geq 0$ and $\sum_\alpha M_\alpha = I$. The probability of obtaining the result α is given by $p(\alpha|x) = \mathrm{Tr}(\rho_x M_\alpha)$. As illustrated in Fig. 1, the measurement changes the parametrized quantum state to the parametrized probability distribution.

The Fisher information metric of the probability distribution is given by the classical Fisher information matrix (CFIM) [25], denoted as $F_C(x)$, whose jkth entry is given by $F_C(x)_{jk} = \int_\alpha [1/p(\alpha|x)][\partial p(\alpha|x)/\partial x_j][\partial p(\alpha|x)/\partial x_k] d\alpha$. The classical Fisher information metric is related to the Euclidean distance between $\sqrt{p(\alpha|x)}$ and $\sqrt{p(\alpha|x+dx)}$ as

$$\int_\alpha (\sqrt{p(\alpha|x)} - \sqrt{p(\alpha|x+dx)})^2 d\alpha = \frac{1}{4} dx F_C(x) dx^T. \qquad (1)$$

We note that for diagonal quantum states, where the diagonal entries can be regarded as the classical probability distribution, the Bures distance reduces to the Euclidean distance.

The QFIM and CFIM characterize the geometrical structure of the parametrized quantum states and the classical probability distribution, respectively. As measurements can be regarded as a special set of quantum channels (known as the quantum-classical channel), the data process inequality tells us that the distance cannot increase under the measurements. This implies that under any measurement $F_C(x) \leq F_Q(x)$.

In classical estimation, the CFIM provides an asymptotically achievable lower bound on the covariance for locally unbiased estimators of the parameters, which is known as the Cramér-Rao bound [26] with $\mathrm{Cov}(\hat{x}) \geq (1/\nu) F_C^{-1}(x)$, where $\hat{x}$ denotes the locally unbiased estimator of x and $\mathrm{Cov}(\hat{x})$ denotes the covariance matrix where the jkth entry is given by $\mathrm{Cov}(\hat{x})_{jk} = E[(\hat{x}_j - x_j)(\hat{x}_k - x_k)]$, ν is the number of sampled data. Since $F_C(x) \leq F_Q(x)$, the covariance matrix is further bounded by the QFIM as $\mathrm{Cov}(\hat{x}) \geq (1/\nu) F_Q^{-1}(x)$, which is known as the quantum Cramér-Rao bound (QCRB) [1,2].

If there exists a measurement such that $F_C(x) = F_Q(x)$, the measurement then preserves the local geometrical structure and the QCRB is saturable. It is known that if the quantum state is parametrized by a single parameter, i.e., $n = 1$, then there always exists a measurement that can preserve the local Fisher information structure [1]. Furthermore, the measurement that saturates the QCRB can be taken as a 1-local measurement; collective measurements are not required. One such measurement is the projective measurement on the eigenvectors of the SLD [1]. When there are multiple parameters, the information structure becomes much more complicated. First, the SLDs for different parameters may not commute with each other; thus in general there does not exist a measurement that can make $F_C(x) = F_Q(x)$ [6,27–62]. The distortion of the Fisher information structure is then typically inevitable. Second, collective measurements matter. As illustrated in Fig. 2, if we repeat the 1-local measurement on two copies of quantum states, the maximal CFIM at most doubles. However, if collective measurements can be performed, it is possible to obtain larger CFIM due to more degrees of freedom in the measurements. In general, under p-local measurements, the CFIM that can be extracted from p copies of quantum states, which we denote as $F_{Cp}(x)$, can be larger than $pF_C(x)$. The CFIM is in general super-additive with respect to the number of copies of quantum states. This is related to the notion of the nonlocality without entanglement [63].

We use $\mathrm{Tr}[F_{Qp}^{-1}(x)F_{Cp}(x)]$ to quantify the difference between the QFIM and the CFIM that can be extracted by p-local measurements on $\rho_x^{\otimes p}$ [27–29,56]. Compared to other quantifiers of the difference, such as $\|F_{Qp}(x)-F_{Cp}(x)\|$, $\mathrm{Tr}[F_{Qp}^{-1}(x)F_{Cp}(x)]$ has the advantage that it is invariant under reparametrization. Since $F_{Cp}(x) \le F_{Qp}(x)$, we have a trivial upper bound $\mathrm{Tr}[F_{Qp}^{-1}(x)F_{Cp}(x)] \le \mathrm{Tr}(I_n) = n$ (here I_n is the $n \times n$ identity matrix), and this upper bound is saturated when there exists a p-local measurement that makes $F_{Cp}(x) = F_{Qp}(x)$. In general, $\mathrm{Tr}[F_{Qp}^{-1}(x)F_{Cp}(x)] < n$ and $n - \mathrm{Tr}[F_{Qp}^{-1}(x)F_{Cp}(x)]$ quantifies the gap between the QFIM and the achievable CFIM under p-local measurements. We use Γ_p to denote the maximal $\mathrm{Tr}[F_{Qp}^{-1}(x)F_{Cp}(x)]$ over all p-local measurements, then $\Gamma_1 \le \Gamma_2 \le \cdots \le \Gamma_\infty \le n$.

Previous results on achievable CFIM are mostly on the extreme cases with $p = 1$, 2 or $p = \infty$ [2,27,29–31,51]. Some of the previous results are stated in terms of the covariance matrix instead of the CFIM; we note that since the classical Cramér-Rao bound is achievable asymptotically, the covariance matrix and the inverse of the CFIM are interchangeable.

For 1-local measurement ($p = 1$), Nagaoka provided a bound for the estimation of two parameters ($n = 2$) as [64,65]

$$\nu \mathrm{Tr}[\mathrm{Cov}(\hat{x})] \ge \min_{\{X_1,X_2\}} \mathrm{Tr}(\rho_x X_1^2) + \mathrm{Tr}(\rho_x X_2^2) + \|\sqrt{\rho_x}[X_1, X_2]\sqrt{\rho_x}\|_1, \tag{2}$$

where $\{X_j\}$ are Hermitian operators that satisfy the locally unbiased condition, $\mathrm{Tr}(\rho_x X_j){=}0$, and $\mathrm{Tr}[(\partial\rho_x/\partial x_k)X_j]{=}\delta_{jk}$, with δ_{jk} as the Kronecker delta function. Recently the Nagaoka bound has been generalized to n parameters [66], which in general can only be evaluated numerically. Gill and Massar provided an analytical bound under 1-local measurements as $\mathrm{Tr}[F_Q^{-1}(x)F_C(x)] \le d - 1$, with d as the dimension of the Hilbert space for a single ρ_x [27]. The Gill-Massar bound is nontrivial only when $n \ge d$.

A necessary condition for the saturation of the QCRB under 1-local measurements is the partial commutative condition [59], which requires the SLDs commute on the support of ρ_x. Specifically, if we write ρ_x in the eigenvalue decomposition as $\rho_x = \sum_{i=1}^m \lambda_i |\Psi_i\rangle\langle\Psi_i|$, with $\lambda_i > 0$, the partial commutative condition is $\langle\Psi_r|[L_j, L_k]|\Psi_s\rangle = 0$ for any $j, k \in \{1, \ldots, n\}$ and $r, s \in \{1, \ldots, m\}$ [59].

For $p = 2$, Zhu and Hayashi provided an upper bound on Γ_2 as $\mathrm{Tr}[F_{Q2}^{-1}(x)F_{C2}(x)] \le \frac{3}{2}(d-1)$ [51], which is nontrivial only when $n \ge \frac{3}{2}(d-1)$.

For $p = \infty$, Holevo provided an achievable bound [2] in terms of the weighted covariance matrix as $\nu \mathrm{Tr}[W\mathrm{Cov}(\hat{x})] \ge \min_{\{X_j\}}\{\mathrm{Tr}[W\mathrm{Re}Z(X)] + \|\sqrt{W}\mathrm{Im}Z(X)\sqrt{W}\|_1\}$, where $W \ge 0$ is a weighted matrix, and $Z(X)$ is a matrix with its jkth entry given by $Z(X)_{jk} = \mathrm{Tr}(\rho_x X_j X_k)$. Here $\{X_1, \ldots, X_n\}$ is a set of Hermitian operators that satisfy the local unbiased condition, and $\mathrm{Re}Z(x)$ and $\mathrm{Im}Z(X)$ are the real and imaginary part of $Z(x)$, respectively. The Holevo bound in general can only be evaluated numerically [62]. For pure states the Holevo bound can be saturated by 1-local measurements [52], while for mixed states the saturation of the Holevo bound in general requires collective measurements on infinite number of copies of the states. The necessary and sufficient condition for the Holevo bound to coincide with the QCRB is the weak commutative condition, which is $\mathrm{Tr}(\rho_x[L_j, L_k]) = 0$ for all $j, k \in \{1, \ldots, n\}$.

For general p-local measurements, there is little understanding on the achievable CFIM. We present an approach that can lead to various bounds Γ_p. These bounds provide a necessary condition for the saturation of the QCRB under general p-local measurements, which recovers the partial commutative condition at $p = 1$ and the weak commutative condition at $p \to \infty$.

For a state ρ_x, with $x = (x_1, \ldots, x_n)$, given any POVM $\{M_\alpha\}$ and any $|u\rangle$, we define Cov_u as an $n \times n$ matrix with the jkth entry given by

$$(\mathrm{Cov}_u)_{jk} = \sum_\alpha [\hat{x}_j(\alpha) - x_j][\hat{x}_k(\alpha) - x_k]\langle u|\sqrt{\rho_x}M_\alpha\sqrt{\rho_x}|u\rangle, \tag{3}$$

and A_u as an $n \times n$ matrix with the jkth entry given by

$$\begin{aligned}(A_u)_{jk} &= \langle u|\sqrt{\rho_x}X_j X_k\sqrt{\rho_x}|u\rangle \\ &= \frac{1}{2}\langle u|\sqrt{\rho_x}\{X_j, X_k\}\sqrt{\rho_x}|u\rangle \\ &\quad + i\frac{1}{2i}\langle u|\sqrt{\rho_x}[X_j, X_k]\sqrt{\rho_x}|u\rangle,\end{aligned} \tag{4}$$

where $\{\hat{x}_j\}$ are locally unbiased estimators and $\{X_j = \sum_\alpha[\hat{x}_j(\alpha) - x_j]M_\alpha\}$ satisfy the locally unbiased condition. For any $|u\rangle$ we can prove that $\mathrm{Cov}_u \ge A_u$ and $\mathrm{Cov}_u \ge A_u^T$. And for any set of $\{|u_q\rangle\}$ that satisfies $\sum_q |u_q\rangle\langle u_q| = I$, it is easy to verify that $\mathrm{Cov}(\hat{x}) = \sum_q \mathrm{Cov}_{u_q}$. Then for any choices of $\bar{\mathbf{A}}_{u_q} \in \{A_{u_q}, A_{u_q}^T\}$, we have $\mathrm{Cov}(\hat{x}) = \sum_q \mathrm{Cov}_{u_q} \ge \bar{\mathbf{A}} = \sum_q \bar{\mathbf{A}}_{u_q}$, where each $\bar{\mathbf{A}}_{u_q}$ can be taken independently as either A_{u_q} or $A_{u_q}^T$. By decomposing $\bar{\mathbf{A}}$ into the real and imaginary part as $\bar{\mathbf{A}} = \bar{\mathbf{A}}_{\mathrm{Re}} + i\bar{\mathbf{A}}_{\mathrm{Im}}$, we obtain a bound on the weighted covariance matrix:

$$\nu \mathrm{Tr}[W\mathrm{Cov}(\hat{x})] \ge \min_{\{X_j\}} \mathrm{Tr}[W\bar{\mathbf{A}}_{\mathrm{Re}}] + \|\sqrt{W}\bar{\mathbf{A}}_{\mathrm{Im}}\sqrt{W}\|_1, \tag{5}$$

where the number of repetitions ν is included. Any choices of $\{|u_q\rangle\}$ with $\sum_q |u_q\rangle\langle u_q| = I$ and any $\bar{\mathbf{A}}_{u_q} \in \{A_{u_q}, A_{u_q}^T\}$ lead to a valid bound.

This provides a versatile tool to obtain many useful bounds by properly choosing $\{|u_q\rangle\}$ and $\{\bar{\mathbf{A}}_{u_q}\}$. In particular, the Holevo bound [2] and the Nagaoka bound [64,65] can be recovered from this general bound by making particular choices of $\{|u_q\rangle\}$ and $\{\bar{\mathbf{A}}_{u_q}\}$ [67]. Furthermore, by combining with an improved Robertson's uncertainty relation for multiple observables [67–69], we can obtain a set of analytical bounds on the gap between the QFIM and the CFIM under general p-local measurements. Specifically for pure states we have

$$\Gamma \leq n - f(n)\|F_Q^{-1/2}F_{\mathrm{Im}}F_Q^{-1/2}\|_F^2, \tag{6}$$

where F_{Im} is the matrix with the entries given by $(F_{\mathrm{Im}})_{jk} = (1/2i)\mathrm{Tr}(\rho_x[L_j, L_k])$ and $\|\cdot\|_F = \sqrt{\sum_{j,k}|(\cdot)_{jk}|^2}$ is the Frobenius norm, $f(n)$ can take $1/4(n-1)$, $(n-2)/(n-1)^2$, or $\frac{1}{5}$, which all lead to valid bounds. Since larger $f(n)$ leads to tighter bound, we can take $f(n) = \max\{1/4(n-1), (n-2)/(n-1)^2, \frac{1}{5}\}$. Note that here we use Γ instead of Γ_p, since for pure states $\Gamma_1 = \Gamma_2 = \cdots = \Gamma_\infty$.

For mixed states under p-local measurements we have

$$\Gamma_p \leq n - f(n)\left\|\frac{F_Q^{-1/2}\bar{\mathbf{F}}_{\mathrm{Im}p}F_Q^{-1/2}}{p}\right\|_F^2, \tag{7}$$

where $f(n) = \max\{1/4(n-1), (n-2)/(n-1)^2, \frac{1}{5}\}$, $\bar{\mathbf{F}}_{\mathrm{Im}p}$ is the imaginary part of $\bar{\mathbf{F}} = \sum_q \bar{F}_{u_q}$ with each $\bar{F}_{u_q}$ equal to either F_{u_q} or $F_{u_q}^T$, where F_{u_q} is an $n\times n$ matrix with the jkth entry given by $(F_{u_q})_{jk} = \langle u_q|\sqrt{\rho_x^{\otimes p}}L_{jp}L_{kp}\times\sqrt{\rho_x^{\otimes p}}|u_q\rangle$, L_{jp} is the SLD of $\rho_x^{\otimes p}$ corresponding to the parameter x_j, $\{|u_q\rangle\}$ are any set of vectors in $H_d^{\otimes p}$ that satisfies $\sum_q|u_q\rangle\langle u_q| = I_{d^p}$ with I_{d^p} denoting the $d^p\times d^p$ identity matrix.

We can also obtain additional bounds by combining different choices of $\{|u_q\rangle\}$. In particular, for mixed states under p-local measurements we can get

$$\Gamma_p \leq n - \frac{1}{4(n-1)}\left\|\frac{C_p}{p}\right\|_F^2, \tag{8}$$

where C_p is an $n\times n$ matrix with the jkth entry given by $(C_p)_{jk} = \frac{1}{2}\|\sqrt{\rho_x^{\otimes p}}[\tilde{L}_{jp}, \tilde{L}_{kp}]\sqrt{\rho_x^{\otimes p}}\|_1$, where $\tilde{L}_{jp}(\tilde{L}_{kp})$ is the SLD for $\rho_{\tilde{x}}^{\otimes p}$ with respect to the parameter $\tilde{x}_j(\tilde{x}_k)$, where $\tilde{x} = F_Q^{1/2}x$ is a reparametrization under which $\tilde{F}_Q = I$. Note that we always have $\|(C_p/p)\|_F \geq \|(F_Q^{-1/2}\bar{\mathbf{F}}_{\mathrm{Im}p}F_Q^{-1/2}/p)\|_F$; the bound in Eq. (8) is thus tighter than the bound in Eq. (7) when $f(n) = 1/4(n-1)$, but it could be less tight when $f(n) = (n-2)/(n-1)^2$ or $\frac{1}{5}$.

The bound in Eq. (8) provides a necessary condition for the saturation of the QCRB under p-local measurements, which is $C_p/p = 0$. For $p = 1$, this condition is equivalent to the partial commutative condition. For $p\to\infty$, we prove that [67]

$$\lim_{p\to\infty}\frac{(C_p)_{jk}}{p} = \frac{1}{2}|\mathrm{Tr}(\rho_x[\tilde{L}_j, \tilde{L}_k])|. \tag{9}$$

At $p\to\infty$ the condition is thus equivalent to the weak commutative condition, $\mathrm{Tr}(\rho_x[\tilde{L}_j, \tilde{L}_k]) = 0,\ \forall\ j, k$. This builds a bridge between the partial commutative condition and the weak commutative condition at the two extremes.

The bound in Eq. (8) involves operators on p copies of quantum states, whose dimension grows exponentially with p. We provide another simpler bound which only uses operators on a single ρ_x as

$$\Gamma_p \leq n - \frac{1}{4(n-1)}\left\|\frac{T_p}{p}\right\|_F^2, \tag{10}$$

where T_p can be computed from the eigenvalues, eigenvectors, and SLDs of a single ρ_x. Specifically, given $\rho_x = \sum_{i=1}^m \lambda_i|\Psi_i\rangle\langle\Psi_i|$ in the eigenvalue decomposition with $\lambda_i > 0$, the jkth entry of T_p is given by

$$(T_p)_{jk} = \frac{1}{2}E\left(\left|\sum_{r=1}^p\langle\Phi_r|[\tilde{L}_j, \tilde{L}_k]|\Phi_r\rangle\right|\right), \tag{11}$$

where $E(\cdot)$ denotes the expected value, and each $|\Phi_r\rangle$ is randomly and independently chosen from the eigenvectors of ρ_x with the probability equal to the corresponding eigenvalue; i.e., each $|\Phi_r\rangle$ takes $|\Psi_i\rangle$ with probability λ_i, $i\in\{1,\ldots,m\}$. $\tilde{L}_j(\tilde{L}_k)$ is the SLD for $\rho_{\tilde{x}}$ with respect to the parameter $\tilde{x}_j(\tilde{x}_k)$, where $\tilde{x} = F_Q^{1/2}x$. The difference between this bound and the bound in Eq. (8) is at most $O(1/\sqrt{p})$. Thus, when the bound in Eq. (8) is hard to compute at large p, we can use this bound instead which is almost as tight.

The upper bounds on Γ_p can be directly transformed to the lower bounds on the covariance matrix. For example, suppose a p-local measurement is repeated with μ times (so total $\nu = \mu p$ copies of ρ_x), from the classical Cramér-Rao bound we have $\mathrm{Cov}(\hat{x}) \geq (1/\mu)F_{Cp}^{-1}(x)$ (here the equality is achievable since the classical Cramér-Rao bound is saturable). This implies that $(1/\mu)\mathrm{Cov}^{-1}(\hat{x}) \leq F_{Cp}(x)$. Any upper bound, $\Gamma_p \leq D$, then leads to an upper bound on $(1/\nu)\mathrm{Tr}[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})]$ as $(1/\nu)\mathrm{Tr}[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})] \leq \mathrm{Tr}(F_{Qp}^{-1}F_{Cp}) \leq D$ (note $F_{Qp} = pF_Q$). From the Cathy-Schwartz inequality, $\mathrm{Tr}[W\mathrm{Cov}(\hat{x})]\mathrm{Tr}[F_Q^{-1}\mathrm{Cov}^{-1}(\hat{x})] \geq (\mathrm{Tr}\sqrt{F_Q^{-1/2}WF_Q^{-1/2}})^2$, we then obtain

$$\nu\mathrm{Tr}[W\mathrm{Cov}(\hat{x})] \geq \frac{(\mathrm{Tr}\sqrt{F_Q^{-1/2}WF_Q^{-1/2}})^2}{D}. \tag{12}$$

By substituting D with any of the upper bounds $\mathrm{Tr}(F_Q^{-1}F_{CP})$ obtained above, we then get analytical bounds on the weighted covariance matrix.

In summary, we provided a framework to quantify the difference between the quantum and classical Fisher information metric under hierarchical quantum measurements. The framework provides a systematic way to generate bounds on the achievable CFIM for general quantum states under general measurements, which significantly improves our understanding on the Fisher information geometry under hierarchical quantum measurements. A necessary condition for the zero gap between the quantum and classical Fisher metric has also been identified, which is shown to recover the partial commutative condition at $p = 1$ and the weak commutative condition at $p \to \infty$. The result can be directly transformed to the precision limits in multiparameter quantum metrology and have implications in various other fields [18,33,70]. The detailed derivation can be found in the companion paper [67], which also contains additional bounds that can be obtained with the framework.

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[1] Carl W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[2] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[3] Samuel L. Braunstein, Carlton M. Caves, and G. J. Milburn, Generalized uncertainty relations: Theory, examples, and Lorentz invariance, Ann. Phys. (N.Y.) **247**, 135 (1996).
[4] Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone, Quantum-enhanced measurements: Beating the standard quantum limit, Science **306**, 1330 (2004).
[5] Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone, Quantum Metrology, Phys. Rev. Lett. **96**, 010401 (2006).
[6] *Asymptotic Theory of Quantum Statistical Inference*, edited by M. Hayashi (World Scientific, Singapore, 2005).
[7] O. E. Barndorff-Nielsen and R. D. Gill, Fisher information in quantum statistics, J. Phys. A **33**, 4481 (2000).
[8] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology, Nat. Phys. **7**, 406 (2011).
[9] Tomohisa Nagata, Ryo Okamoto, Jeremy L. O'brien, Keiji Sasaki, and Shigeki Takeuchi, Beating the standard quantum limit with four-entangled photons, Science **316**, 726 (2007).
[10] B. L. Higgins, D. W. Berry, S. D. Bartlett, H. M. Wiseman, and G. J. Pryde, Entanglement-free Heisenberg-limited phase estimation, Nature (London) **450**, 393 (2007).
[11] G. Y. Xiang, B. L. Higgins, D. W. Berry, H. M. Wiseman, and G. J. Pryde, Entanglement-enhanced measurement of a completely unknown optical phase, Nat. Photonics **5**, 43 (2011).
[12] Sergei Slussarenko, Morgan M. Weston, Helen M. Chrzanowski, Lynden K. Shalm, Varun B. Verma, Sae Woo Nam, and Geoff J. Pryde, Unconditional violation of the shot-noise limit in photonic quantum metrology, Nat. Photonics **11**, 700 (2017).
[13] Shakib Daryanoosh, Sergei Slussarenko, Dominic W. Berry, Howard M. Wiseman, and Geoff J. Pryde, Experimental optical phase measurement approaching the exact Heisenberg limit, Nat. Commun. **9**, 4606 (2018).
[14] Haidong Yuan and Chi-Hang Fred Fung, Quantum parameter estimation with general dynamics, npj Quantum Inf. **3**, 14 (2017).
[15] Haidong Yuan and Chi-Hang Fred Fung, Quantum metrology matrix, Phys. Rev. A **96**, 012310 (2017).
[16] Jing Liu, Mao Zhang, Hongzhen Chen, Lingna Wang, and Haidong Yuan, Optimal scheme for quantum metrology, Adv. Quantum Technol. **5**, 2100080 (2022).
[17] Shun-ichi Amari and Horishi Nagaoka, *Methods of Information Geometry* (Oxford University Press, New York, 2000).
[18] Johannes Jakob Meyer, Fisher Information in noisy intermediate-scale quantum applications, Quantum **5**, 539 (2021).
[19] Shi-Jian Gu, Fidelity approach to quantum phase transitions, Int. J. Mod. Phys. B **24**, 4371 (2010).
[20] Teng-Long Wang, Ling-Na Wu, Wen Yang, Guang-Ri Jin, Neill Lambert, and Franco Nori, Quantum Fisher information as a signature of the superradiant quantum phase transition, New J. Phys. **16**, 063039 (2014).
[21] Philipp Hyllus, Wiesław Laskowski, Roland Krischek, Christian Schwemmer, Witlef Wieczorek, Harald Weinfurter, Luca Pezzé, and Augusto Smerzi, Fisher information and multiparticle entanglement, Phys. Rev. A **85**, 022321 (2012).
[22] Géza Tóth, Multipartite entanglement and high-precision metrology, Phys. Rev. A **85**, 022322 (2012).
[23] James Martens, New insights and perspectives on the natural gradient method, J. Mach. Learn. Res. **21**, 1 (2020), https://jmlr.org/papers/v21/17- 678.html.
[24] D. J. C. Bures, An extension of Kakutani's theorem on infinite product measures to tensor product of semifinite w^*-algebras, Trans. Am. Math. Soc. **135**, 199 (1969).
[25] R. A. Fisher, On the mathematical foundations of theoretical statistics, Phil. Trans. R. Soc. A **222**, 309 (1922).
[26] Harald Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, NJ, 1946).
[27] Richard D. Gill and Serge Massar, State estimation for large ensembles, Phys. Rev. A **61**, 042312 (2000).
[28] F. Albarelli, M. Barbieri, M. G. Genoni, and I. Gianani, A perspective on multiparameter quantum metrology: From theoretical tools to applications in quantum imaging, Phys. Lett. A **384**, 126311 (2020).
[29] Angelo Carollo, Bernardo Spagnolo, Alexander A. Dubkov, and Davide Valenti, On quantumness in multi-parameter quantum estimation, J. Stat. Mech. (2019) 094010.
[30] Xiao-Ming Lu and Xiaoguang Wang, Incorporating Heisenberg's Uncertainty Principle into Quantum Multiparameter Estimation, Phys. Rev. Lett. **126**, 120503 (2021).
[31] Sammy Ragy, Marcin Jarzyna, and Rafal Demkowicz-Dobrzański, Compatibility in multiparameter quantum metrology, Phys. Rev. A **94**, 052108 (2016).

[32] Yu Chen and Haidong Yuan, Maximal quantum Fisher information matrix, New J. Phys. **19**, 063023 (2017).
[33] Jing Liu, Haidong Yuan, Xiao-Ming Lu, and Xiaoguang Wang, Quantum Fisher information matrix and multiparameter estimation, J. Phys. A **53**, 023001 (2020).
[34] Hongzhen Chen and Haidong Yuan, Optimal joint estimation of multiple Rabi frequencies, Phys. Rev. A **99**, 032122 (2019).
[35] Francesco Albarelli, Marco Barbieri, Marco G. Genoni, and Ilaria Gianani, A perspective on multiparameter quantum metrology: From theoretical tools to applications in quantum imaging, Phys. Lett. A **384**, 126311 (2020).
[36] R. Demkowicz-Dobrzański, W. Górecki, and M. Guţă, Multi-parameter estimation beyond quantum Fisher information, J. Phys. A **53**, 363001 (2020).
[37] Jasminder S. Sidhu and Pieter Kok, Geometric perspective on quantum parameter estimation, AVS Quantum Sci. **2**, 014701 (2020).
[38] M. D. Vidrighin, G. Donati, M. G. Genoni, X.-M. Jin, W. S. Kolthammer, M. S. Kim, A. Datta, M. Barbieri, and I. A. Walmsley, Joint estimation of phase and phase diffusion for quantum metrology, Nat. Commun. **5**, 3532 (2014).
[39] Philip J. D. Crowley, Animesh Datta, Marco Barbieri, and I. A. Walmsley, Tradeoff in simultaneous quantum-limited phase and loss estimation in interferometry, Phys. Rev. A **89**, 023845 (2014).
[40] J.-D. Yue, Y.-R. Zhang, and H. Fan, Quantum-enhanced metrology for multiple phase estimation with noise, Sci. Rep. **4**, 5933 (2014).
[41] Yu-Ran Zhang and Heng Fan, Quantum metrological bounds for vector parameters, Phys. Rev. A **90**, 043818 (2014).
[42] Jing Liu and Haidong Yuan, Control-enhanced multiparameter quantum estimation, Phys. Rev. A **96**, 042114 (2017).
[43] Emanuele Roccia, Ilaria Gianani, Luca Mancino, Marco Sbroscia, Fabrizia Somma, Marco G. Genoni, and Marco Barbieri, Entangling measurements for multiparameter estimation with two qubits, Quantum Sci. Technol. **3**, 01LT01 (2017).
[44] Sholeh Razavian, Matteo G. A. Paris, and Marco G. Genoni, On the quantumness of multiparameter estimation problems for qubit systems, Entropy **22**, 1197 (2020).
[45] Alessandro Candeloro, Matteo G. A. Paris, and Marco G. Genoni, On the properties of the asymptotic incompatibility measure in multiparameter quantum estimation, J. Phys. A **54**, 485301 (2021).
[46] Koichi Yamagata, Akio Fujiwara, and Richard D. Gill, Quantum local asymptotic normality based on a new quantum likelihood ratio, Ann. Stat. **41**, 2197 (2013).
[47] Jonas Kahn and Mdlin Gu, Local asymptotic normality for finite dimensional quantum systems, Commun. Math. Phys. **289**, 597 (2009).
[48] Yuxiang Yang, Giulio Chiribella, and Masahito Hayashi, Attaining the ultimate precision limit in quantum state estimation, Commun. Math. Phys. **368**, 223 (2019).
[49] Jun Suzuki, Explicit formula for the Holevo bound for two-parameter qubit-state estimation problem, J. Math. Phys. (N.Y.) **57**, 042201 (2016).
[50] Jasminder S. Sidhu, Yingkai Ouyang, Earl T. Campbell, and Pieter Kok, Tight Bounds on the Simultaneous Estimation of Incompatible Parameters, Phys. Rev. X **11**, 011028 (2021).
[51] Huangjun Zhu and Masahito Hayashi, Universally Fisher-Symmetric Informationally Complete Measurements, Phys. Rev. Lett. **120**, 030404 (2018).
[52] K. Matsumoto, A new approach to the Cramér-Rao-type bound of the pure-state model, J. Phys. A **35**, 3111 (2002).
[53] Peter C. Humphreys, Marco Barbieri, Animesh Datta, and Ian A. Walmsley, Quantum Enhanced Multiple Phase Estimation, Phys. Rev. Lett. **111**, 070403 (2013).
[54] Luca Pezzè, Mario A. Ciampini, Nicolò Spagnolo, Peter C. Humphreys, Animesh Datta, Ian A. Walmsley, Marco Barbieri, Fabio Sciarrino, and Augusto Smerzi, Optimal Measurements for Simultaneous Quantum Estimation of Multiple Phases, Phys. Rev. Lett. **119**, 130504 (2017).
[55] Zhibo Hou, Zhao Zhang, Guo-Yong Xiang, Chuan-Feng Li, Guang-Can Guo, Hongzhen Chen, Liqiang Liu, and Haidong Yuan, Minimal Tradeoff and Ultimate Precision Limit of Multiparameter Quantum Magnetometry under the Parallel Scheme, Phys. Rev. Lett. **125**, 020501 (2020).
[56] Federico Belliardo and Vittorio Giovannetti, Incompatibility in quantum parameter estimation, New J. Phys. **23**, 063055 (2021).
[57] Zhibo Hou, Jun-Feng Tang, Hongzhen Chen, Haidong Yuan, Guo-Yong Xiang, Chuan-Feng Li, and Guang-Can Guo, Zero–trade-off multiparameter quantum estimation via simultaneously saturating multiple Heisenberg uncertainty relations, Sci. Adv. **7**, eabd2986 (2021).
[58] Zhibo Hou, Yan Jin, Hongzhen Chen, Jun-Feng Tang, Chang-Jiang Huang, Haidong Yuan, Guo-Yong Xiang, Chuan-Feng Li, and Guang-Can Guo, "Super-Heisenberg" and Heisenberg Scalings Achieved Simultaneously in the Estimation of a Rotating Field, Phys. Rev. Lett. **126**, 070503 (2021).
[59] Jing Yang, Shengshi Pang, Yiyu Zhou, and Andrew N. Jordan, Optimal measurements for quantum multiparameter estimation with general states, Phys. Rev. A **100**, 032104 (2019).
[60] Nan Li, Christopher Ferrie, Jonathan A. Gross, Amir Kalev, and Carlton M. Caves, Fisher-Symmetric Informationally Complete Measurements for Pure States, Phys. Rev. Lett. **116**, 180402 (2016).
[61] Magdalena Szczykulska, Tillmann Baumgratz, and Animesh Datta, Multi-parameter quantum metrology, Adv. Phys. **1**, 621 (2016).
[62] Francesco Albarelli, Jamie F. Friel, and Animesh Datta, Evaluating the Holevo Cramér-Rao Bound for Multiparameter Quantum Metrology, Phys. Rev. Lett. **123**, 200503 (2019).
[63] Charles H. Bennett, David P. DiVincenzo, Christopher A. Fuchs, Tal Mor, Eric Rains, Peter W. Shor, John A. Smolin, and William K. Wootters, Quantum nonlocality without entanglement, Phys. Rev. A **59**, 1070 (1999).
[64] H. Nagaoka, A new approach to Cramér-Rao bounds for quantum state estimation, in *Asymptotic Theory of Quantum Statistical Inference: Selected Papers*, edited by Masahito Hayashi (World Scientific, Singapore, 2005), originally

published as IEICE Technical Report, 89, 228, IT 8942, 914 (1989).

[65] H. Nagaoka, A generalization of the simultaneous diagonalization of Hermitian matrices and its relation to quantum estimation theory, in *Asymptotic Theory of Quantum Statistical Inference: Selected Papers* (World Scientific, Singapore, 2005).

[66] Lorcn O. Conlon, Jun. Suzuki, Ping Koy Lam, and Syed M. Assad, Efficient computation of the Nagaoka-Hayashi bound for multiparameter estimation with separable measurements, npj Quantum Inf. **7**, 110 (2021).

[67] Hongzhen Chen, Yu Chen, and Haidong Yuan, companion paper, Incompatibility measures in multiparameter quantum estimation under hierarchical quantum measurements, Phys. Rev. A **105** 062442 (2022).

[68] H. P. Robertson, An indeterminacy relation for several observables and its classical interpretation, Phys. Rev. **46**, 794 (1934).

[69] D. A. Trifonov, Generalizations of Heisenberg uncertainty relation, Eur. Phys. J. B **29**, 349 (2002).

[70] Tobias Haug, Kishor Bharti, and M. S. Kim, Capacity and quantum geometry of parametrized quantum circuits, PRX Quantum **2**, 040309 (2021).