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Simultaneous measurement of multiple incompatible observables and tradeoff in multiparameter quantum estimation

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How well can multiple incompatible observables be implemented by a single measurement? This is a fundamental problem in quantum mechanics with wide implications for the performance optimization of numerous tasks in quantum information science. While existing studies have been mostly focusing on the approximation of two observables with a single measurement, in practice multiple observables are often encountered, for which the errors of the approximations are little understood. Here we provide a framework to study the implementation of an arbitrary finite number of observables with a single measurement. Our methodology yields novel analytical bounds on the errors of these implementations, significantly advancing our understanding of this fundamental problem. Additionally, we introduce a more stringent bound utilizing semi-definite programming that, in the context of two observables, generates an analytical bound tighter than previously known bounds. The derived bounds have direct applications in assessing the trade-off between the precision of estimating multiple parameters in quantum metrology, an area with crucial theoretical and practical implications. To validate the validity of our findings, we conducted experimental verification using a superconducting quantum processor. This experimental validation not only confirms the theoretical results but also effectively bridges the gap between the derived bounds and empirical data obtained from real-world experiments. Our work paves the way for optimizing various tasks in quantum information science that involve multiple noncommutative observables.

One of the distinctive features of quantum mechanics is its noncommutativity, setting it apart from classical physics. This noncommutativity is prominently manifested in the properties of observables, leading to phenomena that defy classical expectations. When multiple observables commute with each other, their simultaneous measurement is feasible through projective measurements on their shared eigenspaces. However, for noncommuting observables, exact simultaneous measurement becomes unattainable, necessitating approximation. A central issue in understanding and harnessing the full potential of quantum systems is then determining the degree to which a single measurement can accurately capture multiple non-commuting observables. Intuitively, measuring multiple noncommuting observables comes with an inherent tradeoff. As we strive to estimate one observable with higher accuracy, the imprecision in determining other incompatible observables tends to increase. This tradeoff is deeply rooted in the uncertainty principle¹, which stands as a fundamental principle in quantum mechanics.

The standard Robertson-Schrödinger uncertainty relation^{2,3}, $(\Delta X_1)^2(\Delta X_2)^2 \geq \frac{1}{4}|\text{Tr}[\rho[X_1, X_2]]|^2$ with $(\Delta X_{1/2})^2 = \langle X_{1/2}^2 \rangle - \langle X_{1/2} \rangle^2$, describes the impossibility of preparing quantum states with sharp distributions for non-commuting observables simultaneously, which is also referred to as the preparation uncertainty relation. The uncertainty relations that describe the approximation of non-commuting observables via a single measurement are called measurement uncertainty relations^{4–18}. In the field of measurement uncertainty relations, there are two main approaches: the state-independent approach^{17–22}, which establish error bounds for measurement devices regardless of the input state; and state-dependent relations^{4–15}, which assess the trade-off between errors in joint measurements on a predetermined quantum state. In this article, we focus on state-dependent measurement uncertainty relations as they are particularly relevant to multi-parameter quantum estimation where the state is often constrained.

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Current research in state-dependent measurement uncertainty relations predominantly concentrate on the error-tradeoff relations for approximating pairs of observables^{6–15}, yet practical applications typically require dealing with multiple observables. Fields such as vector magnetometry^{23,24}, reference frame alignment²⁵ and quantum imaging^{26,27} all require a nuanced understanding and manipulation of three or more observables simultaneously. The inherent uncertainty therein cannot be fully understood or quantified through the lens of pairwise uncertainty relations. This gap highlights a critical need for expanding the framework to encompass multiple observables, which is of both fundamental and practical importance. However, similar to numerous challenges in quantum information science, akin to the complex quantification of multipartite entanglement²⁸, extending results from two-party scenarios to multi-partite ones often necessitates novel methodologies.

In this article, we present dual methods capable of yielding error-tradeoff relations for approximating an arbitrary number of observables. The first approach delivers analytical tradeoff relations for any number of observables, while the second method offers more stringent bounds through semidefinite programming. By merging these strategies, we are able to derive analytical tradeoff relations that are even tighter than any existing trade-off relations for two observables. Subsequently, we apply these approaches to quantum metrology, deriving tighter tradeoff relations for estimating an arbitrary number of parameters—a topic central to contemporary quantum estimation research. Furthermore, we empirically validate our findings through experimentation conducted on a superconducting quantum processor. Our results provide significant insights into the interplay among multiple observables involved in various quantum information tasks, notably in the calibration of performance for multiparameter quantum metrology.

Results

Analytical error-tradeoff relation

We commence by deriving an analytical measurement uncertainty relation for a general set of n observables. The objective is to use a single Positive Operator-Valued Measurement (POVM), denoted $\mathcal{M} = \{M_m\}$, to approximate the given n observables X_1, X_2, \dots, X_n when applied to a quantum state ρ and to determine relations that set limits on the minimum cumulative weighted approximation error.

According to Neumark’s dilation theorem²⁹, the POVM, $\mathcal{M} = \{M_m = K_m^\dagger K_m\}$, is equivalent to a projective measurement on $\rho \otimes \sigma$ in an extended Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_A$, here $\sigma = |\xi_0\rangle\langle\xi_0|$ is an ancillary state such that $(I \otimes \langle\xi_0|)U^\dagger(I \otimes |\xi_m\rangle\langle\xi_m|)U(I \otimes |\xi_0\rangle) = M_m$, where $\{|\xi_m\rangle\}$ is an orthonormal basis for the ancillary system, U is a unitary operator on the extended space such that for any $|\psi\rangle$, $U|\psi\rangle \otimes |\xi_0\rangle = \sum_m K_m |\psi\rangle |\xi_m\rangle$. Denote $V_m = U^\dagger(I \otimes |\xi_m\rangle\langle\xi_m|)U$, we then have $\text{Tr}[(\rho \otimes |\xi_0\rangle\langle\xi_0|)V_m] = \text{Tr}(\rho M_m)$. From the measurement, we can construct a set of commuting observables, $\{F_j = \sum_m f_j(m)V_m | 1 \leq j \leq n\}$, to approximate $\{X_j \otimes I\}$ in the extended Hilbert space (see Fig. 1). The mean

squared error of the approximation on the state is given by^{6,7,12}

$$\epsilon_j^2 = \text{Tr}[(F_j - X_j \otimes I)^2(\rho \otimes \sigma)]. \tag{1}$$

In the case of two observables, Ozawa obtained an error-tradeoff relation as^{6,7}

$$\epsilon_1 \cdot \epsilon_2 + \epsilon_1 \cdot \Delta X_2 + \Delta X_1 \cdot \epsilon_2 \geq c_{12}, \tag{2}$$

here $\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}$ is the standard deviation of an observable on the state with $\langle \cdot \rangle = \text{Tr}(\rho \cdot)$, $c_{12} = \frac{1}{2} \text{Tr}(\rho[X_1, X_2])$. Branciard strengthened this relation as^{12,13}

$$\epsilon_1^2 \cdot (\Delta X_2)^2 + (\Delta X_1)^2 \cdot \epsilon_2^2 + 2\sqrt{(\Delta X_1)^2 \cdot (\Delta X_2)^2 - c_{12}^2} \epsilon_1 \cdot \epsilon_2 \geq c_{12}^2, \tag{3}$$

which is tight for pure states. For mixed states, Ozawa further tightened the relation by replacing c_{12} with $\frac{1}{2} \|\sqrt{\rho}[X_1, X_2]\sqrt{\rho}\|_1$ ¹⁵. However, it is worth noting that even with this improvement, for mixed states the bound is not tight, and the geometrical method employed to derive these relations are not readily extendable to scenarios involving more than two observables. For general n observables, the error-tradeoff relation is little understood.

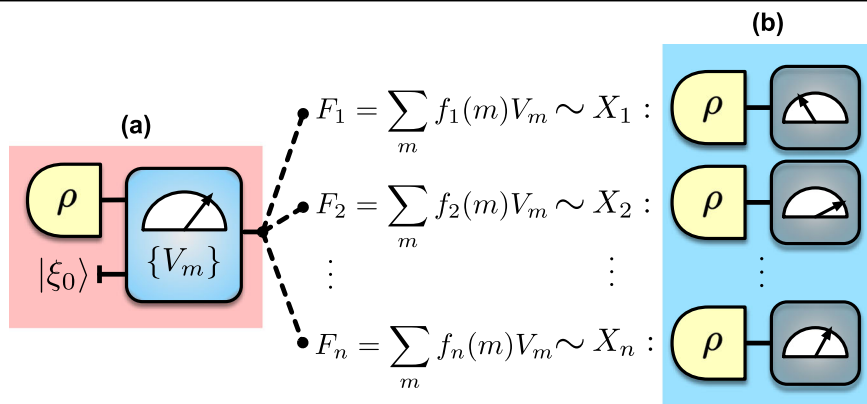
We now present an approach that can lead to analytical tradeoff relations for an arbitrary number of observables. Let

$$\mathcal{A}_u = \begin{pmatrix} \langle u | \sqrt{\rho} \otimes \sigma E_1 \\ \vdots \\ \langle u | \sqrt{\rho} \otimes \sigma E_n \\ \langle u | \sqrt{\rho} \otimes \sigma (X_1 \otimes I) \\ \vdots \\ \langle u | \sqrt{\rho} \otimes \sigma (X_n \otimes I) \end{pmatrix} \begin{pmatrix} \langle u | \sqrt{\rho} \otimes \sigma E_1 \\ \vdots \\ \langle u | \sqrt{\rho} \otimes \sigma E_n \\ \langle u | \sqrt{\rho} \otimes \sigma (X_1 \otimes I) \\ \vdots \\ \langle u | \sqrt{\rho} \otimes \sigma (X_n \otimes I) \end{pmatrix}^\dagger = \begin{pmatrix} Q_u & R_u \\ R_u^\dagger & S_u \end{pmatrix} \geq 0, \tag{4}$$

here $E_j = F_j - X_j \otimes I$ is the error operator, $|u\rangle$ is any vector, Q_u, R_u, S_u are $n \times n$ matrices with the entries given by (note E_j and X_j are all Hermitian operators)

$$\begin{aligned} (Q_u)_{jk} &= \langle u | \sqrt{\rho} \otimes \sigma E_j E_k \sqrt{\rho} \otimes \sigma | u \rangle, \\ (R_u)_{jk} &= \langle u | \sqrt{\rho} \otimes \sigma E_j (X_k \otimes I) \sqrt{\rho} \otimes \sigma | u \rangle, \\ (S_u)_{jk} &= \langle u | \sqrt{\rho} \otimes \sigma (X_j \otimes I) (X_k \otimes I) \sqrt{\rho} \otimes \sigma | u \rangle. \end{aligned} \tag{5}$$

Fig. 1 | Simultaneous measurement of multiple observables $\{X_1, X_2, \dots, X_n\}$ via a single measurement. As illustrated in **a**, a Positive Operator-Valued Measure (POVM) applied to ρ effectively acts as a projective measurement $\{V_m\}$ on the extended state $\rho \otimes |\xi_0\rangle\langle\xi_0|$. This setup enables the construction of a set of commuting observables $\{F_1, F_2, \dots, F_n\}$, as depicted in **b**, designed to approximate the individual measurements of each observable in the set $\{X_1, X_2, \dots, X_n\}$.



We can write these matrices in terms of the real and imaginary parts as $Q_u = Q_{u,\text{Re}} + iQ_{u,\text{Im}}, R_u = R_{u,\text{Re}} + iR_{u,\text{Im}}, S_u = S_{u,\text{Re}} + iS_{u,\text{Im}}$.

Given any set of states $\{|u_q\rangle\}$ such that $\sum_q |u_q\rangle\langle u_q| = I$, we can derive a corresponding set of matrices $\{\tilde{\mathcal{A}}_{u_q}\}$. We then construct a matrix \tilde{A} as the sum of these matrices with each $\tilde{\mathcal{A}}_{u_q}$ being either \mathcal{A}_{u_q} or its transpose, $\mathcal{A}_{u_q}^T$. Since both \mathcal{A}_{u_q} and $\mathcal{A}_{u_q}^T$ are positive semi-definite, it follows that:

$$\tilde{A} = \sum_q \tilde{\mathcal{A}}_{u_q} = \begin{pmatrix} \tilde{Q} & \tilde{R} \\ \tilde{R}^\dagger & \tilde{S} \end{pmatrix} \geq 0, \tag{6}$$

where the components are defined as $\tilde{Q} = \sum_q \tilde{Q}_{u_q}, \tilde{R} = \sum_q \tilde{R}_{u_q}$, and $\tilde{S} = \sum_q \tilde{S}_{u_q}$, with every $(\tilde{Q}_{u_q}, \tilde{R}_{u_q}, \tilde{S}_{u_q})$ being either $(Q_{u_q}, R_{u_q}, S_{u_q})$ or their complex conjugate $(\bar{Q}_{u_q}, \bar{R}_{u_q}, \bar{S}_{u_q})$, here $\tilde{M} = M_{\text{Re}} - iM_{\text{Im}}$ and for Hermitian matrix $\tilde{M} = M^T$. It's important to highlight that the real parts of the matrix elements in \tilde{A} are unaffected by the choice between \mathcal{A}_{u_q} and its transpose. Consequently, the real parts of \tilde{Q} and \tilde{S} remain constant and are determined solely by the original components without regard to whether they were chosen as Q_{u_q} or \bar{Q}_{u_q} (and correspondingly for S_{u_q}), with their specific values given by

$$\begin{aligned} (\tilde{Q}_{\text{Re}})_{jk} &= \sum_q (\tilde{Q}_{u_q, \text{Re}})_{jk} = \frac{1}{2} \text{Tr}[(\rho \otimes \sigma)\{E_j, E_k\}], \\ (\tilde{S}_{\text{Re}})_{jk} &= \sum_q (\tilde{S}_{u_q, \text{Re}})_{jk} = \frac{1}{2} \text{Tr}(\rho\{X_j, X_k\}). \end{aligned} \tag{7}$$

Specifically, the diagonal elements of \tilde{Q} and \tilde{S} are given as $(\tilde{Q})_{jj} = \epsilon_j^2$ and $(\tilde{S})_{jj} = \text{Tr}(\rho X_j^2)$, respectively.

From Eq. (6), we can derive an analytical error-tradeoff relation for approximating n observables (see Section S1 of the Supplementary Material for a detailed derivation):

$$\text{Tr}(S_{\text{Re}}^{-1} Q_{\text{Re}}) \geq \left(\sqrt{\|S_{\text{Re}}^{-\frac{1}{2}} \tilde{S}_{\text{Im}} S_{\text{Re}}^{-\frac{1}{2}}\|_F + 1} - 1 \right)^2, \tag{8}$$

where $\|\cdot\|_F = \sqrt{\sum_{j,k} |(\cdot)_{jk}|^2}$ represents the Frobenius norm. In this inequality, the term Q_{Re} is the sole quantity dependent on the measurement strategy and its diagonal entries correspond to the mean-square errors of the approximation. Both S_{Re} and \tilde{S}_{Im} are independent of the specific measurement process; instead, they are entirely determined by the inherent properties of the observables when applied to the given quantum state.

The inequality in Eq. (8) establishes a fundamental limit on the minimum achievable errors for any POVM that approximates the given set of observables on a quantum state. It provides a bound that holds true for any choice of orthonormal basis $|u_q\rangle$, and the tightest bound can be obtained by optimizing over all possible $|u_q\rangle$.

In the case of pure states, the selection of a specific $|u_q\rangle$ is not necessary. The derived analytical bound guarantees to be tighter than simply summing up Branciard's bounds for two observables pairwise when the total number of observables exceeds four. The comparison is detailed in Section S7 of the Supplementary Material.

It is also important to recognize that Eq. (6) inherently implies $\tilde{S} \geq 0$, which constitutes a refined version of Robertson's preparation uncertainty relation that solely reflects the observables' properties on the state without considering any measurements. The conventional Robertson's preparation uncertainty can be viewed as a specific instance of this refinement by consistently setting $\tilde{S}_{u_q} = S_{u_q}$. This refined formulation thus also paves the way for tighter preparation uncertainty bounds with independent significance.

Error-tradeoff relation via semidefinite programming

We proceed to introduce a secondary approach that yields even tighter tradeoff relations. This method bypasses the need for selecting specific $\{|u_q\rangle\}$ and can be formulated as semi-definite programming (SDP), enabling efficient computation through readily available algorithms such as CVX^{30,31} and YALMIP³².

Again for any POVM, $\{M_m\} \in H_S$, it can be realized as projective measurement, $\{V_m\} \in H_S \otimes H_A$, with $(I \otimes \langle \xi_0 |) V_m (I \otimes |\xi_0 \rangle) = M_m$. We can then construct $\{F_j = \sum_m f_j(m) V_m\}$ to approximate $\{X_j \otimes I_A\}$. Let Q be an $n \times n$ Hermitian matrix, with its jk th element given as

$$\begin{aligned} Q_{jk} &= \text{Tr}[(\rho \otimes \sigma)(F_j - X_j \otimes I)(F_k - X_k \otimes I)] \\ &= \text{Tr}\left[\rho \sum_m f_j(m) M_m f_k(m)\right] - \text{Tr}(\rho R_j X_k) \\ &\quad - \text{Tr}(\rho X_j R_k) + \text{Tr}(\rho X_j X_k), \end{aligned} \tag{9}$$

here $R_j = (I \otimes \langle \xi_0 |) F_j (I \otimes |\xi_0 \rangle) = \sum_m f_j(m) M_m$ is a Hermitian matrix in \mathcal{H}_S . We let \mathbb{S} be a $n \times n$ block operator whose jk -th block is $\mathbb{S}_{jk} = \sum_m f_j(m) M_m f_k(m)$, which is itself a Hermitian matrix, and let $\mathbb{R} = (R_1 \ R_2 \ \dots \ R_n)^\dagger, \mathbb{X} = (X_1 \ X_2 \ \dots \ X_n)^\dagger$. \mathbb{S} and \mathbb{R} both depend on the measurement with $\mathbb{S}_{jk} = \mathbb{S}_{jk}^\dagger = S_{kj}, R_j = R_j^\dagger$. We have $\mathbb{S} \geq \mathbb{R} \mathbb{R}^\dagger$ (see Section S2 of the Supplementary Material). Q can then be rewritten as $Q = \text{Tr}_{\mathbb{S}}[(I_n \otimes \rho)(\mathbb{S} - \mathbb{R} \mathbb{X}^\dagger - \mathbb{X} \mathbb{R}^\dagger + \mathbb{X} \mathbb{X}^\dagger)]$ and the weighted mean squared error can be written as

$$\mathcal{E} = \text{Tr}(WQ) = \text{Tr}[(W \otimes \rho)(\mathbb{S} - \mathbb{R} \mathbb{X}^\dagger - \mathbb{X} \mathbb{R}^\dagger + \mathbb{X} \mathbb{X}^\dagger)], \tag{10}$$

where $W \geq 0$ is a weighted matrix, which typically takes a diagonal form as $W = \text{diag}\{w_1, \dots, w_n\}$, but can also take other forms.

Now assume $\{R_j\}_{j=1}^n$ and \mathbb{S}^* are the optimal operators that lead to the minimal error, we then have

$$\begin{aligned} \mathcal{E} &\geq \text{Tr}[(W \otimes \rho)(\mathbb{S}^* - \mathbb{R}^* \mathbb{X}^\dagger - \mathbb{X} \mathbb{R}^{*\dagger} + \mathbb{X} \mathbb{X}^\dagger)] \\ &\geq \min_{\mathbb{S}, \{R_j\}_{j=1}^n} \left\{ \text{Tr}[(W \otimes \rho)(\mathbb{S} - \mathbb{R} \mathbb{X}^\dagger - \mathbb{X} \mathbb{R}^\dagger + \mathbb{X} \mathbb{X}^\dagger)] \right. \\ &\quad \left. | \mathbb{S} \geq \mathbb{R} \mathbb{R}^\dagger, \mathbb{S}_{jk} = S_{kj} = S_{jk}^\dagger, R_j = R_j^\dagger \right\}. \end{aligned} \tag{11}$$

The minimization can be formulated as a semi-definite programming with

$$\begin{aligned} \mathcal{E}_0 &= \min_{\mathbb{S}, \{R_j\}_{j=1}^n} \text{Tr}[(W \otimes \rho)(\mathbb{S} - \mathbb{R} \mathbb{X}^\dagger - \mathbb{X} \mathbb{R}^\dagger + \mathbb{X} \mathbb{X}^\dagger)] \\ &\text{subject to} \quad \mathbb{S}_{jk} = S_{kj} = S_{jk}^\dagger, \forall j, k \\ &\quad R_j = R_j^\dagger, \forall j \\ &\quad \begin{pmatrix} I & \mathbb{R}^\dagger \\ \mathbb{R} & \mathbb{S} \end{pmatrix} \geq 0. \end{aligned} \tag{12}$$

The derived lower bound, $\mathcal{E} \geq \mathcal{E}_0$, offers a tighter constraint than the analytical bounds from the previous section for any selection of $\{|u_q\rangle\}$ (refer to Section S3 of the Supplementary Material). Furthermore, an explicit construction detailing the optimal approximation strategy that attains this bound for pure states is provided in Section S4 of the Supplementary Material, which demonstrates the tightness of the bound for any number of observables when applied to pure states. For mixed states, however, the bound is in general not tight (see Section S4 of the Supplementary Material for an example).

Tighter analytical relation for two observables

By leveraging the SDP bound provided in Eq. (12) and employing a judicious selection of $|u_q\rangle$ analogs to the analytical bound in Eq. (8), we can derive analytical bounds on mixed states for two observables that are

tighter than the Ozawa’s relation¹⁵, the tightest analytical bound previously known.

We first show that (see Section S5 of the Supplementary Material for details) when $\rho = |\psi\rangle\langle\psi|$ is a pure state and $W = \text{diag}\{w_1, w_2\}$, Eq. (12) can be analytically solved as

$$w_1\epsilon_1^2 + w_2\epsilon_2^2 \geq \frac{1}{2} \left(\alpha - \sqrt{\alpha^2 - \beta^2} \right), \tag{13}$$

where

$$\begin{aligned} \alpha &= w_1(\Delta X_1)^2 + w_2(\Delta X_2)^2, \\ \beta &= i\sqrt{w_1 w_2} \langle \psi | [X_1, X_2] | \psi \rangle. \end{aligned} \tag{14}$$

Since the SDP bound is tight for pure state, this analytical bound is also tight for pure states. We now use it to obtain tighter analytical bounds for two observables on mixed states. For a mixed state, ρ , we can choose any $\{|u_q\rangle\}$ with $\sum_q |u_q\rangle\langle u_q| = I$ and write

$$\rho = \sum_q \sqrt{p} |u_q\rangle\langle u_q| \sqrt{p} = \sum_q \lambda_q |\phi_q\rangle\langle\phi_q|, \tag{15}$$

here $\lambda_q = \langle u_q | \rho | u_q \rangle$, $|\phi_q\rangle = \frac{\sqrt{p}|u_q\rangle}{\sqrt{\langle u_q | \rho | u_q \rangle}}$. For each $|\phi_q\rangle$ we can get a corresponding lower bound $\mathcal{E}_{|\phi_q\rangle}$ by substituting $|\phi_q\rangle\langle\phi_q|$ in Eq. (12), and solve it analytically to get $\mathcal{E}_{|\phi_q\rangle} = \frac{1}{2} (\alpha_q - \sqrt{\alpha_q^2 - \beta_q^2})$, where α_q and β_q are obtained from Eq. (14) by substituting $|\psi\rangle$ with $|\phi_q\rangle$. Since for any function that satisfies $f(\sum_q \lambda_q x_q, y) = \sum_q \lambda_q f(x_q, y)$, we have $\min_y f(\sum_q \lambda_q x_q, y) \geq \sum_q \lambda_q \min_y f(x_q, y)$, by substitute f with $\text{Tr}[(W \otimes \rho)(S - \mathbb{R}\mathbb{X}^\dagger - \mathbb{X}\mathbb{R}^\dagger + \mathbb{X}\mathbb{X}^\dagger)]$, x_q with $|\phi_q\rangle\langle\phi_q|$ and $\sum_q \lambda_q x_q$ with ρ in Eq. (12), we can get $\mathcal{E}_0 \geq \sum_q \lambda_q \mathcal{E}_{|\phi_q\rangle}$. This then leads to an analytical bound

$$w_1\epsilon_1^2 + w_2\epsilon_2^2 \geq \mathcal{E}_0 \geq \sum_q \lambda_q \mathcal{E}_{|\phi_q\rangle} = \sum_q \frac{\lambda_q}{2} \left(\alpha_q - \sqrt{\alpha_q^2 - \beta_q^2} \right) \equiv \mathcal{E}_A. \tag{16}$$

In comparison, the minimal weighted error corresponding to the Ozawa’s relation is given by (see Section S6 of the Supplementary Material)

$$w_1\epsilon_1^2 + w_2\epsilon_2^2 \geq \frac{1}{2} \left(\alpha_\rho - \sqrt{\alpha_\rho^2 - \beta_\rho^2} \right) \equiv \mathcal{E}_{\text{Ozawa}}, \tag{17}$$

with $\alpha_\rho = w_1(\Delta X_1)^2 + w_2(\Delta X_2)^2$ and $\beta_\rho = i\sqrt{w_1 w_2} \|\sqrt{\rho}[X_1, X_2]\sqrt{\rho}\|$, here $(\Delta X_{1/2})^2 = \text{Tr}(\rho X_{1/2}^2) - \text{Tr}(\rho X_{1/2})^2$.

By choosing $\{|u_q\rangle\}$ as the eigenstates of $\sqrt{\rho}[X_1, X_2]\sqrt{\rho}$, we can get (see Section S6 of the Supplementary Material for detail)

$$\sum_q \frac{\lambda_q}{2} \left(\alpha_q - \sqrt{\alpha_q^2 - \beta_q^2} \right) \geq \frac{1}{2} \left(\alpha_\rho - \sqrt{\alpha_\rho^2 - \beta_\rho^2} \right). \tag{18}$$

The analytical bound in Eq. (16) is thus tighter than the bound obtained from the Ozawa’s relation. An illustrative example is shown in Fig. 2 with

$$\rho = \begin{pmatrix} \frac{p}{2} & 0 & 0 \\ 0 & 1-p & 0 \\ 0 & 0 & \frac{p}{2} \end{pmatrix}, X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}. \tag{19}$$

The presented framework thus not only extends to scenarios involving an arbitrary number of observables but also provides improved analytical bounds in the case of two observables.

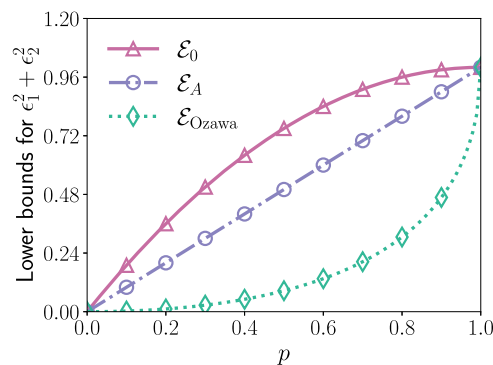


Fig. 2 | Lower bounds for the errors of the simultaneous measurement of two observables X_1, X_2 on state ρ with different p , where X_1, X_2 , and ρ are given in Eq. (19). Here \mathcal{E}_0 is computed with SDP in Eq. (12), \mathcal{E}_A is analytically obtained from Eq. (16) and $\mathcal{E}_{\text{Ozawa}}$ is obtained from the Ozawa’s relation (Eq. (17)). It can be seen that for the simultaneous measurement of X_1, X_2 , we have $\mathcal{E}_0 \geq \mathcal{E}_A \geq \mathcal{E}_{\text{Ozawa}}$.

Tradeoff relations for multiparameter quantum estimation

We proceed to apply the derived error-tradeoff relations to the field of quantum metrology, thereby elucidating tradeoffs in the precision limits for estimating multiple parameters, which is currently a central focus in quantum metrology.

Given a quantum state ρ_x , where $x = (x_1, x_2, \dots, x_n)$ are n unknown parameters to be estimated, by performing a POVM, $\{M_m\}$, on the state, we can get the measurement result, m , with a probability $p_m(x) = \text{Tr}(\rho_x M_m)$. For any locally unbiased estimator, $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$, the Cramér-Rao bound^{33,34} provides an achievable bound $\text{Cov}(\hat{x}) \geq \frac{1}{\nu} F_C^{-1}$, here $\text{Cov}(\hat{x})$ is the covariance matrix for the estimators with the jk -th entry given by $\text{Cov}(\hat{x})_{jk} = E[(\hat{x}_j - x_j)(\hat{x}_k - x_k)]$, $E[\cdot]$ denotes the expectation, ν is the number of the measurement repeated independently, F_C is the Fisher information matrix of a single measurement whose jk -th entry is given by $(F_C)_{jk} = \sum_m \frac{\partial_{x_j} p_m(x) \partial_{x_k} p_m(x)}{p_m(x)}$ ³⁴. Regardless of the choice of the measurement, the covariance matrix is always lower bounded by the quantum Cramér-Rao bound as^{35,36}

$$\text{Cov}(\hat{x}) \geq \frac{1}{\nu} F_C^{-1} \geq \frac{1}{\nu} F_Q^{-1}, \tag{20}$$

here F_Q is the quantum Fisher information matrix with the jk -th entry given by $(F_Q)_{jk} = \text{Tr}(\rho_x \frac{L_j L_k + L_k L_j}{2})$, where L_q is the symmetric logarithmic derivative (SLD) corresponding to the parameter x_q which satisfies $\partial_{x_q} \rho_x = \frac{1}{2} (\rho_x L_q + L_q \rho_x)$. When there is only one parameter, the QCRB can be saturated. In particular it can be saturated with the projective measurement on the eigen-spaces of the SLD, i.e., the SLD is the optimal observable for the estimation of the corresponding parameter.

When there are multiple parameters, the QCRB is in general not saturable since the SLDs typically do not commute with each other. A central task in multi-parameter quantum estimation is to understand the tradeoff induced by such incompatibility^{23,27,37-70}. In a recent seminal work⁴⁶, by applying Ozawa’s uncertainty relation, Lu and Wang obtained analytical tradeoff relations for the estimation of a pair of parameters. For multiple ($n > 2$) parameters, however, if we simply add the tradeoff for each pair of parameters directly, the obtained tradeoff relation is typically loose, which restricts the scope of its applications.

Analytical tradeoff relation for multiparameter quantum estimation.

By directly applying the tradeoff relation from Eq. (8), we can readily obtain a tradeoff relation for estimating multiple parameters by simply substituting the n observables with the n SLDs, $\{L_1, \dots, L_n\}$. In this context, S_{Re} corresponds to the quantum Fisher information matrix, F_Q . For any

POVM $\{M_m\}$, we construct $\{F_j = \sum_m f_j(m) V_m\}$ to approximate the SLDs. As previously defined, each V_m represents a projective measurement in the extended space that projects onto M_m when acting on the system. Given that the error-tradeoff relation in Eq. (8) holds for any choice of the functions $\{f_j(m)\}$, we can specifically select

$$f_j(m) = \frac{1}{2} \frac{\text{Tr}[\{L_j \otimes I, \rho_x \otimes \sigma\} V_m]}{\text{Tr}[\rho_x \otimes \sigma] V_m} = \frac{\partial_{x_j} p_m(x)}{p_m(x)}, \quad (21)$$

where $p_m(x) = \text{Tr}[(\rho_x \otimes \sigma) V_m] = \text{Tr}[\rho_x M_m]$. This choice minimizes $\epsilon_j^2 = \text{Tr}[(F_j - L_j \otimes I)^2 (\rho_x \otimes \sigma)]$ under the given measurement (see Methods). With this choice we have $Q_{\text{Re}} = F_Q - F_C$, Eq. (8) then becomes

$$\text{Tr}[F_Q^{-1}(F_Q - F_C)] \geq \left(\sqrt{\|F_Q^{-\frac{1}{2}} \tilde{S}_{\text{Im}} F_Q^{-\frac{1}{2}}\|_F} + 1 - 1 \right)^2, \quad (22)$$

where $\tilde{S}_{\text{Im}} = \sum_q \tilde{S}_{u_q, \text{Im}}$ with each $\tilde{S}_{u_q, \text{Im}}$ equals to either $S_{u_q, \text{Im}}$ or $S_{u_q, \text{Im}}^T$, here $(S_{u_q, \text{Im}})_{jk} = \frac{1}{2i} \langle u_q | (\sqrt{\rho_x} \otimes |\xi_0\rangle \langle \xi_0|) (L_j, L_k) \otimes D(\sqrt{\rho_x} \otimes |\xi_0\rangle \langle \xi_0|) |u_q\rangle$ with $\{|u_q\rangle\}$ as any set of vectors that satisfies $\sum_q |u_q\rangle \langle u_q| = I$. This then provides an upper bound on the achievable classical Fisher information matrix as

$$\text{Tr}[F_Q^{-1} F_C] \leq n - \left(\sqrt{\|F_Q^{-\frac{1}{2}} \tilde{S}_{\text{Im}} F_Q^{-\frac{1}{2}}\|_F} + 1 - 1 \right)^2. \quad (23)$$

In comparison with the previous metrological bounds in refs. 42,43, the current bound is less stringent. This is because the previous metrological bounds exploit the properties of locally unbiased estimators, which introduce additional constraints that are not present in the simultaneous measurement of multiple observables. The bound here is based only on the characteristics of the SLD observables, reflecting the inherent uncertainties of the observables. The difference in constraints is what creates the difference in strictness between the previous bounds and the current one. This also means that previous metrological bounds cannot be directly generalized to the uncertainty relations for incompatible observables.

SDP-based tradeoff relation for multiparameter quantum estimation. In the scenario of estimating multiple parameters encapsulated in a quantum state, ρ_x , with $x = (x_1, \dots, x_n)$, we can substitute the matrix \mathbb{X} in Eq. (12) with $\tilde{\mathbb{L}} = (L_1 \ L_2 \ \dots \ L_n)^\dagger$, here L_q is the SLD for the parameter x_q . Upon choosing the optimal functions $f_j(m)$, the real part of the error matrix becomes $Q_{\text{Re}} = F_Q - F_C$. The derived bound $\mathcal{E} \geq \mathcal{E}_0$ then gives a tradeoff relation in multiparameter quantum estimation

$$\mathcal{E} = \text{Tr}(WQ) = \text{Tr}(WQ_{\text{Re}}) = \text{Tr}[W(F_Q - F_C)] \geq \mathcal{E}_0, \quad (24)$$

where we used the property that $\text{Tr}(WQ_{\text{Im}}) = \text{Tr}(W^{\frac{1}{2}} Q_{\text{Im}} W^{\frac{1}{2}}) = 0$ since Q_{Im} is anti-symmetric, and here

$$\begin{aligned} \mathcal{E}_0 &= \min_{S, \{R_j\}_{j=1}^n} \text{Tr}[(W \otimes \rho_x)(S - \mathbb{R}\mathbb{L}^\dagger - \mathbb{L}\mathbb{R}^\dagger + \mathbb{L}\mathbb{L}^\dagger)] \\ \text{subject to} \quad & S_{jk} = S_{kj} = S_{jk}^\dagger, \forall j, k \\ & R_j = R_j^\dagger, \forall j \\ & \begin{pmatrix} I & \mathbb{R}^\dagger \\ \mathbb{R} & S \end{pmatrix} \geq 0. \end{aligned} \quad (25)$$

We can perform a reparameterization by setting $\tilde{x} = F_Q^{-\frac{1}{2}} x$ which leads to $\tilde{F}_Q = I$. Under this transformation, we have that $\text{Tr}[I \otimes \rho_x] \tilde{\mathbb{L}} \tilde{\mathbb{L}}^\dagger] = \text{Tr}(\tilde{F}_Q) = n$, where $\tilde{\mathbb{L}} = (\tilde{L}_1 \ \tilde{L}_2 \ \dots \ \tilde{L}_n)^\dagger$ with $\tilde{L}_j = \sum_k (F_Q^{-\frac{1}{2}})_{jk} L_k$. By setting $W = I$ we can derive an upper bound for

$$\text{Tr}(\tilde{F}_C) = \text{Tr}(F_Q^{-1} F_C)$$

$$\begin{aligned} \text{Tr}(F_Q^{-1} F_C) &\leq \max_{S, \{R_j\}_{j=1}^n} \text{Tr}[(I \otimes \rho_x)(-S + \mathbb{R}\mathbb{L}^\dagger + \mathbb{L}\mathbb{R}^\dagger)] \\ \text{subject to} \quad & S_{jk} = S_{kj} = S_{jk}^\dagger, \forall j, k \\ & R_j = R_j^\dagger, \forall j \\ & \begin{pmatrix} I & \mathbb{R}^\dagger \\ \mathbb{R} & S \end{pmatrix} \geq 0. \end{aligned} \quad (26)$$

This resembles the Nagaoka-Hayashi bound⁵⁰ but are different. The Nagaoka-Hayashi bound quantifies $\text{Tr}[WCov(\hat{x})]$, where \hat{x} is required to be locally unbiased estimators to satisfy the classical Cramér-Rao bound as $Cov(\hat{x}) \geq \frac{1}{\nu} F_C^{-1}$. The Nagaoka-Hayashi bound thus has the locally unbiased condition included in the constraints. And it does not have a direct connection to the approximation of the observables, which is reflected in the fact that its objective function does not contain the SLDs. While the bound here quantifies directly the relation between F_Q and F_C , without the intermediate step of estimators, it thus does not have the locally unbiased condition in the constraints. The presence of the SLD operators in the objective function underscores an intrinsic connection to observable approximation, which is absent in the Nagaoka-Hayashi bound.

Sharper tradeoff relation for two-parameter quantum estimation. In the case of estimating two parameters, a tighter analytical bound can be derived by substituting (X_1, X_2) in Eq. (16) with (L_1, L_2) , which results in

$$\begin{aligned} w_1[(F_Q)_{11} - (F_C)_{11}] + w_2[(F_Q)_{22} - (F_C)_{22}] \\ \geq \sum_q \frac{\lambda_q}{2} \left(\alpha_q - \sqrt{\alpha_q^2 - \beta_q^2} \right), \end{aligned} \quad (27)$$

where $\lambda_q = \langle u_q | \rho_x | u_q \rangle$, α_q and β_q are given as

$$\begin{aligned} \alpha_q &= w_1(\Delta_{|\phi_q\rangle} L_1)^2 + w_2(\Delta_{|\phi_q\rangle} L_2)^2, \\ \beta_q &= i\sqrt{w_1 w_2} \langle \phi_q | [L_1, L_2] | \phi_q \rangle, \end{aligned} \quad (28)$$

here $|\phi_q\rangle = \frac{\sqrt{\rho_x} |u_q\rangle}{\sqrt{\langle u_q | \rho_x | u_q \rangle}}$ and $\Delta_{|\phi_q\rangle} L_{1/2}$ denotes the standard deviation of the SLD $L_{1/2}$ on the state $|\phi_q\rangle$. By selecting $|u_q\rangle$ as the eigenvectors of $\sqrt{\rho_x} [L_1, L_2] \sqrt{\rho_x}$, Eq. (27) provides a tighter bound than the Lu-Wang bound for mixed states⁴⁶, which is based on Ozawa's relation (see Section S6 of the supplementary material for detail).

Experiment validation of the error-tradeoff relations in a superconducting quantum processor

We conducted an experimental verification of the error-tradeoff relations on a superconducting quantum processor, utilizing the Quafu cloud quantum computing platform⁷¹. The selected processor, ScQ-P136, consists of 136 qubits with single-qubit gate fidelities surpassing 99%⁷¹⁻⁷³, and for our analysis, we focused exclusively on the first qubit. Further details about the processor's architecture and parameters are provided in the Methods section.

To experimentally quantify the error ϵ_j for each observable X_j when measured through a specific measurement set $\{M_m\}$, we adopt the "3-state method"^{74,75}. The essence of this approach is illustrated in Fig. 3a, which involves preparing and measuring three distinct quantum states:

$$\rho_1 = \rho, \rho_2 \simeq X_j \rho X_j, \rho_3 \simeq (I + X_j) \rho (I + X_j). \quad (29)$$

By analyzing the measurement statistics of $\{M_m\}$ on these three states ρ_1, ρ_2 , and ρ_3 we can obtain ϵ_j , the error of the approximation. Detailed information on how to determine the error from these measurements can be found in the Methods section.

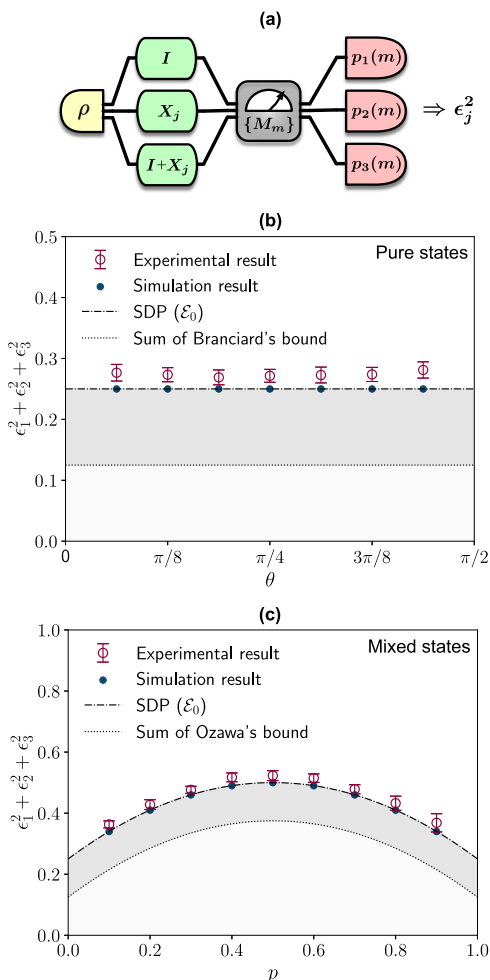


Fig. 3 | Experimental results for testing the error-tradeoff relations in superconducting quantum processor “ScQ-P136” accessed through the Quafu cloud quantum computation platform. **a** Scheme diagram to evaluate the mean squared error of each X_j using the “3-state method”. **b** Error-tradeoff relations for the simultaneous measurement of three spin operators, $\{\frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y, \frac{1}{2}\sigma_z\}$, on a pure state $|\psi\rangle = R_z(\frac{\pi}{2})R_y(\theta)|0\rangle$. **c** Error-tradeoff relations for the simultaneous measurement of three spin operators, $\{\frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y, \frac{1}{2}\sigma_z\}$, on a mixed state $\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$. For both cases, the statistics of each measurement was obtained through 2000 shots and the error bars represent the standard deviations obtained by repeating the experiment 20 times.

We commence with the simultaneous measurement of the three Pauli spin operators, $\{\frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y, \frac{1}{2}\sigma_z\}$, on a single qubit system. The state of the qubit is first prepared as a pure state, specifically a rotation of the ground state by an angle θ around the y -axis followed by a rotation of $\frac{\pi}{2}$ around the z -axis, which can be expressed as $|\psi\rangle = R_z(\frac{\pi}{2})R_y(\theta)|0\rangle$ with $0 < \theta \leq \frac{\pi}{2}$. Before conducting the experiment, we first substitute the given qubit state and spin operators into Eq. (12) to derive a tight theoretical lower bound for the total mean-squared-error $\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2$. The optimal primal variables of \mathcal{E}_0 are recorded as $\{R_1^\dagger, R_2^\dagger, R_3^\dagger\}$. Leveraging these values, we construct the corresponding optimal measurement scheme $\{M_m^*\}$ (see Sec. S4 of the Supplementary Material for details).

In the experimental phase, we apply the optimal measurement scheme $\{M_m^*\}$ to the prepared pure state $\rho = |\psi\rangle\langle\psi|$ to approximate the joint measurement of the three Pauli operators $\{\frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y, \frac{1}{2}\sigma_z\}$. The errors for each operator are then estimated using the “3-state method” by preparing auxiliary states $\rho_2 \simeq X_j\rho X_j$ and $\rho_3 \simeq (I + X_j)\rho(I + X_j)$. We carry out this procedure for various values of the rotation angle θ . The corresponding total mean-squared-errors are plotted in Fig. 3(b), where simulated results are also presented for comparison.

Next, we proceed to verify our tradeoff relations for mixed states by simultaneously measuring the three spin operators $\{\frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y, \frac{1}{2}\sigma_z\}$ on a mixed state of the form $\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$, where $0 \leq p \leq 1$. Similarly, prior to the experimental phase, we input the mixed state and observables into Eq. (12) to derive a theoretical lower bound for the total mean-squared-error $\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2$. This is achieved by solving the SDP problem and computing its optimal solution. In the context of mixed states, we resort to numerical methods to determine the corresponding optimal measurements $\{M_m^*\}$ instead of analytical solutions. These numerically derived optimal measurements are then applied in the experimental setup to approximate the joint measurement of the spin operators $\{\frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y, \frac{1}{2}\sigma_z\}$ on the given mixed state. Once more, we utilize the “3-state method” to calculate the total errors by preparing auxiliary states: $\rho_2 \simeq X_j\rho X_j$ and $\rho_3 \simeq (I + X_j)\rho(I + X_j)$. This process is repeated for a range of different mixing parameter values p , and the corresponding total mean-squared-errors are presented graphically in Fig. 3(c), along with accompanying simulation data for comparison.

Discussion

We have developed methodologies that establish tradeoff relations for approximating any number of observables using a single measurement, and we have also refined the existing analytical bounds in scenarios involving two observables. Each derived bound has its unique benefits and drawbacks:

1. The inequality

$$\text{Tr}(S_{\text{Re}}^{-1}Q_{\text{Re}}) \geq \left(\sqrt{\|S_{\text{Re}}^{-\frac{1}{2}}\tilde{S}_{\text{Im}}S_{\text{Re}}^{-\frac{1}{2}}\|_F} + 1 - 1 \right)^2$$

offers an analytical constraint applicable to any number of observables. For pure states, the selection of $\{|u_q\rangle\}$ is not necessary and the bound outperforms the sum of Ozawa’s relations when estimating more than four observables. However, for mixed states, the tightness depends on the choice of the set $\{|u_q\rangle\}$ where $|u_q\rangle\langle u_q| = I$ (see Section S9.A of the Supplementary Material for an explicit example). Although every selection of $\{|u_q\rangle\}$ produces a valid bound, there is currently no systematic method available for identifying the optimal choice. This analytical expression provides a universal benchmark but might not always yield the most stringent limitation, especially in complex mixed-state scenarios.

2. The error tradeoff relation $\mathcal{E} = \text{Tr}(WQ) \geq \mathcal{E}_0$ provides the most stringent bound, where \mathcal{E}_0 can be efficiently computed using semi-definite programming (SDP),

$$\begin{aligned} \mathcal{E}_0 &= \min_{\mathcal{S}, \{R_j\}_{j=1}^n} \text{Tr}[(W \otimes \rho)(\mathcal{S} - \mathbb{R}\mathbb{X}^\dagger - \mathbb{X}\mathbb{R}^\dagger + \mathbb{X}\mathbb{X}^\dagger)] \\ \text{subject to : } & \mathcal{S}_{jk} = \mathcal{S}_{kj} = \mathcal{S}_{jk}^\dagger, \forall j, k \\ & R_j = R_j^\dagger, \forall j \\ & \begin{pmatrix} I & \mathbb{R}^\dagger \\ \mathbb{R} & \mathcal{S} \end{pmatrix} \geq 0. \end{aligned}$$

This SDP-based approach is universally applicable for any number of observables and offers a tighter bound than the first analytical relation. For pure states, this bound is exact, and it outperforms the first bound for mixed states regardless of the choice of the set $\{|u_q\rangle\}$ satisfying $\sum_q |u_q\rangle\langle u_q| = I$. Furthermore, it consistently provides a stricter constraint compared to the sum of Ozawa’s relations in all scenarios. This SDP-based bound does not have a general analytical expression, necessitating numerical methods for its computation.

3. For a pair of observables X_j and X_k acting on the state ρ , we have an analytical bound given by:

$$w_j\epsilon_j^2 + w_k\epsilon_k^2 \geq \sum_q \frac{\lambda_q}{2} \left(\alpha_q - \sqrt{\alpha_q^2 - \beta_q^2} \right),$$

where $\lambda_q = \langle u_q|\rho|u_q\rangle$, $|\phi_q\rangle = \frac{\sqrt{\rho}|u_q\rangle}{\sqrt{\langle u_q|\rho|u_q\rangle}}$, $\alpha_q = w_j[\langle\phi_q|X_j^2|\phi_q\rangle - \langle\phi_q|X_j|\phi_q\rangle^2] + w_k[\langle\phi_q|X_k^2|\phi_q\rangle - \langle\phi_q|X_k|\phi_q\rangle^2]$, $\beta_q = i\sqrt{w_jw_k}\langle\phi_q|[X_j, X_k]|\phi_q\rangle$. This

inequality holds for any choice of $\{|u_q\rangle\}$ satisfying $\sum_q |u_q\rangle\langle u_q| = I$. Notably, if one selects $\{|u_q\rangle\}$ to be the eigenvectors of $\sqrt{\rho}[X_j, X_k]\sqrt{\rho}$, this bound is tighter than Ozawa’s relation for mixed states. However, it only applies to pairs of observables.

Each of the error-tradeoff relations can be directly applied to assess the precision tradeoffs in multi-parameter quantum metrology. These relations are valid for local measurements, which involve independently measuring each copy of the state ρ_x . When considering p -local measurements, where the measurement process may involve collective actions on up to p copies of ρ_x , analogous tradeoff relations can be derived by substituting ρ_x and its set of SLDs $\{L_j\}$ with $\rho_x^{\otimes p}$ and the corresponding SLDs for $\rho_x^{\otimes p}$. In Section S8 of the Supplementary Materials, we present an illustrative example that showcases the tradeoff relation under collective measurements. This paves the way for further exploration into state-dependent measurement uncertainty relations for multiple observables. Moreover, it reinforces the connection between measurement uncertainty and the incompatibility inherent to multi-parameter quantum estimation, thereby promoting deeper investigations across both domains.

Methods

Tradeoff in multi-parameter quantum estimation

By taking the set of SLDs, $\{L_q\}$, as the observables, we can obtain the trade-off relations in multiparameter quantum estimation.

For any POVM $\mathcal{M} = \{M_m\}$, it can be equivalently written as a projective measurement $\{V_m = U^\dagger(I \otimes |\xi_m\rangle\langle \xi_m|)U\}$ on an extended state $\rho_x \otimes \sigma$ with $\sigma = |\xi_0\rangle\langle \xi_0|$ such that $(I \otimes \langle \xi_0|)V_m(I \otimes |\xi_0\rangle) = M_m$. Commuting observables, $\{F_1, F_2, \dots, F_n\}$ with $F_j = \sum_m f_j(m)V_m$ are then constructed from this measurement on the extended Hilbert space to approximate $\{L_1 \otimes I, \dots, L_n \otimes I\}$. Note that the tradeoff relations for the approximate measurement holds for any choice of $\{f_j(m)\}$, here we make a particular choice of $\{f_j(m)\}$ to minimize the root-mean-squared error $\{\epsilon_j^2 = \text{Tr}[(F_j - L_j \otimes I)^2(\rho_x \otimes \sigma)]\}$. As

$$\begin{aligned} \text{Tr}[(F_j - L_j \otimes I)^2(\rho_x \otimes \sigma)] &= \text{Tr}\left[\left(\sum_m f_j(m)V_m - L_j \otimes I\right)^2(\rho_x \otimes \sigma)\right] \\ &= \text{Tr}(\rho_x L_j^2) - \sum_m p_m(x) \left(\frac{1}{2} \frac{\text{Tr}[(L_j \otimes I, \rho_x \otimes \sigma)V_m]}{\text{Tr}(\rho_x \otimes \sigma)V_m}\right)^2 \\ &\quad + \sum_m p_m(x) \left(f_j(m) - \frac{1}{2} \frac{\text{Tr}[(L_j \otimes I, \rho_x \otimes \sigma)V_m]}{\text{Tr}(\rho_x \otimes \sigma)V_m}\right)^2, \end{aligned} \tag{30}$$

here $p_m(x) = \text{Tr}[(\rho_x \otimes \sigma)V_m] = \text{Tr}(M_m \rho_x)$ is the probability for the measurement result m . The optimal $f_j(m)$ that minimizes the root-mean-squared error is then given by

$$f_j(m) = \frac{1}{2} \frac{\text{Tr}[(L_j \otimes I, \rho_x \otimes \sigma)V_m]}{\text{Tr}[(\rho_x \otimes \sigma)V_m]}. \tag{31}$$

We note that when $p_m(x) = 0$, $f_j(m)$ can take the form as $0/0$, which should be computed as a multivariate limit with $x' \rightarrow x$.

So far we have not used any properties of the SLDs, the formula for the optimal choice of $f_j(m)$ works for any observables¹². For the SLDs in particular, we have

$$\begin{aligned} f_j(m) &= \frac{1}{2} \frac{\text{Tr}[(L_j \otimes I, \rho_x \otimes \sigma)V_m]}{\text{Tr}[(\rho_x \otimes \sigma)V_m]} \\ &= \frac{\text{Tr}[(\frac{1}{2}(L_j \rho_x + \rho_x L_j) \otimes \sigma)V_m]}{p_m(x)} \\ &= \frac{\partial_{x_j} \text{Tr}[(\rho_x \otimes \sigma)V_m]}{p_m(x)} = \frac{\partial_{x_j} p_m(x)}{p_m(x)}. \end{aligned} \tag{32}$$

With this optimal choice, we have $F_j = \sum_m \frac{\partial_{x_j} p_m(x)}{p_m(x)} V_m$. The entries of Q_{Re} can then be obtained as

$$\begin{aligned} (Q_{\text{Re}})_{jk} &= \text{Re Tr}[(F_j - L_j \otimes I)(F_k - L_k \otimes I)(\rho_x \otimes \sigma)] \\ &= \text{Re} \left\{ \text{Tr}[(F_j F_k)(\rho_x \otimes \sigma)] - \text{Tr}[F_j(L_k \otimes I)(\rho_x \otimes \sigma)] \right. \\ &\quad \left. - \text{Tr}[(L_j \otimes I)F_k(\rho_x \otimes \sigma)] + \text{Tr}[(L_j L_k \otimes I)(\rho_x \otimes \sigma)] \right\}. \end{aligned} \tag{33}$$

Here the first term,

$$\begin{aligned} \text{Re} \{ \text{Tr}[F_j F_k(\rho_x \otimes \sigma)] \} &= \text{Re} \left\{ \text{Tr} \left[\sum_m \frac{\partial_{x_j} p_m(x)}{p_m(x)} \frac{\partial_{x_k} p_m(x)}{p_m(x)} V_m(\rho_x \otimes \sigma) \right] \right\} \\ &= \sum_m \frac{\partial_{x_j} p_m(x) \partial_{x_k} p_m(x)}{p_m(x)}, \end{aligned} \tag{34}$$

equals to the jk -th entry of the classical Fisher information matrix, $(F_C)_{jk}$. While the second term,

$$\begin{aligned} \text{Re} \{ \text{Tr}[F_j(L_k \otimes I)(\rho_x \otimes \sigma)] \} &= \frac{1}{2} \text{Tr}[(F_j(L_k \otimes I) + (L_k \otimes I)F_j)(\rho_x \otimes \sigma)] \\ &= \sum_m \frac{\partial_{x_j} p_m(x)}{p_m(x)} \text{Tr}[\frac{1}{2}(L_k \rho_x + \rho_x L_k) \otimes \sigma)V_m] \\ &= \sum_m \frac{\partial_{x_j} p_m(x)}{p_m(x)} \partial_{x_k} \text{Tr}[(\rho_x \otimes \sigma)V_m] \\ &= \sum_m \frac{\partial_{x_j} p_m(x) \partial_{x_k} p_m(x)}{p_m(x)}, \end{aligned} \tag{35}$$

also equals to $(F_C)_{jk}$. It can be similarly shown that the third term equals to $(F_C)_{jk}$ as well. For the last term we have

$$\begin{aligned} \text{Re} \{ \text{Tr}[(L_j L_k \otimes I)(\rho_x \otimes \sigma)] \} &= \text{Re} \{ \text{Tr}[L_j L_k \rho_x] \} \\ &= \frac{1}{2} \text{Tr}[(L_j L_k + L_k L_j) \rho_x], \end{aligned} \tag{36}$$

which is just $(F_Q)_{jk}$. Put the four terms together, we can get $(Q_{\text{Re}})_{jk} = (F_Q)_{jk} - (F_C)_{jk}$. It is also straightforward to see that with the SLDs as the observables, we have $S_{\text{Re}} = F_Q$. The error tradeoff for the approximate measurement of the SLDs then leads to a tradeoff relation

$$\text{Tr} [F_Q^{-1}(F_Q - F_C)] \geq \left(\sqrt{\|F_Q^{-\frac{1}{2}} \tilde{S}_{\text{Im}} F_Q^{-\frac{1}{2}}\|_F + 1} - 1 \right)^2, \tag{37}$$

where $\tilde{S}_{\text{Im}} = \sum_q \tilde{S}_{u_q, \text{Im}}$ with each $\tilde{S}_{u_q, \text{Im}}$ equals to either $S_{u_q, \text{Im}}$ or $S_{u_q, \text{Im}}^T$, here $(S_{u_q, \text{Im}})_{jk} = \frac{1}{2i} \langle u_q | (\sqrt{\rho_x} \otimes |\xi_0\rangle\langle \xi_0|) [L_j, L_k] \otimes I (\sqrt{\rho_x} \otimes |\xi_0\rangle\langle \xi_0|) | u_q \rangle$ with $\{|u_q\rangle\}$ as any set of vectors that satisfies $\sum_q |u_q\rangle\langle u_q| = I$. This can be rewritten as

$$\text{Tr} [F_Q^{-1} F_C] \leq n - \left(\sqrt{\|F_Q^{-\frac{1}{2}} \tilde{S}_{\text{Im}} F_Q^{-\frac{1}{2}}\|_F + 1} - 1 \right)^2. \tag{38}$$

Evaluate the mean-squared-error using “3-state method”

To experimentally evaluate the errors ϵ_j , note that for each j ,

$$\begin{aligned} \epsilon_j^2 &= \text{Tr}[(\rho \otimes \sigma)(F_j - X_j \otimes I)^2] \\ &= \text{Tr}(\rho X_j^2) + \sum_m f_j(m)^2 \text{Tr}(\rho M_m) \\ &\quad - 2 \sum_m f_j(m) \text{Re Tr}(\rho M_m X_j). \end{aligned} \tag{39}$$

Here, $\text{Tr}(\rho M_m)$ represents the probability of obtaining outcome m in the measurement, which can be directly obtained from experimental data. To

evaluate the quantities $\text{ReTr}(\rho M_m X_j)$, the “3-state method” can be employed^{74,75}. We can express the terms as

$$\text{ReTr}(\rho M_m X_j) = \frac{1}{2} \left(\text{Tr} \left[M_m (I + X_j) \rho (I + X_j) \right] - \text{Tr} \left(M_m X_j \rho X_j \right) - \text{Tr} \left(M_m \rho \right) \right). \quad (40)$$

By performing the measurement M_m on three different states $\rho_1 = \rho$, $\rho_2 = \frac{X_j \rho X_j}{\text{Tr}(X_j \rho X_j)}$ and $\rho_3 = \frac{(I+X_j)\rho(I+X_j)}{\text{Tr}[(I+X_j)\rho(I+X_j)]}$, we can estimate the three terms in the expression above. Assuming that all three terms are non-zero without loss of generality, we can calculate the optimal value for $f_j(m)$ as $f_j(m) = \frac{\text{ReTr}(\rho M_m X_j)}{\text{Tr}(\rho M_m)}$. This value can be directly computed using the obtained values of $\text{Tr}(\rho M_m)$ and $\text{ReTr}(\rho M_m X_j)$.

We generalize the method in the previous work⁷⁵ to bound the value of $\text{ReTr}(\rho M_m X_j)$ from noisy experimental data. Specifically, by taking the probabilities $p_l(m) = \text{Tr}(\rho_l M_m)$ $l=1,2,3$ as constraints on the POVM, M_m , we can bound the value of $\text{ReTr}(\rho M_m X_j)$ within a small interval by com-

puting its minimum and maximum, which can be efficiently computed via semi-definite programmings as

$$\begin{aligned} & \min / \max \text{Re Tr}(\rho M_m X_j) \\ & \text{subject to } \sum_m M_m = I, M_m \geq 0, \forall m \\ & \mathcal{S}(\{\text{Tr}(\rho_l M_m) = p_l(m), \forall m | l = 1, 2, 3\}) \end{aligned} \quad (41)$$

here $\mathcal{S}(\mathcal{A})$ denotes the largest subset of \mathcal{A} such that all the constraints therein are independent.

Using the method described, we can evaluate the errors for each observable using the following equations:

$$\begin{aligned} \alpha_{jm}^{\min / \max} &= \min / \max \text{ReTr}(\rho M_m X_j), \\ f_j^{\min / \max}(m) &= \alpha_{jm}^{\min / \max} / p_1(m), \\ (e_j^A)^2 &= \text{Tr}(\rho X_j^2) + \sum_m f_j^{\min}(m)^2 p_1(m) \\ &\quad - 2 \sum_m f_j^{\min}(m) \alpha_{jm}^{\min}, \\ &= \text{Tr}(\rho X_j^2) - \sum_m \left(\alpha_{jm}^{\min} \right)^2 / p_1(m) \\ (e_j^B)^2 &= \text{Tr}(\rho X_j^2) + \sum_m f_j^{\max}(m)^2 p_1(m) \\ &\quad - 2 \sum_m f_j^{\max}(m) \alpha_{jm}^{\max}, \\ &= \text{Tr}(\rho X_j^2) - \sum_m \left(\alpha_{jm}^{\max} \right)^2 / p_1(m). \end{aligned} \quad (42)$$

Denoting $\epsilon_j^{\min / \max} = \min / \max \{e_j^A, e_j^B\}$, the error ϵ_j is then bounded in a small interval $[\epsilon_j^{\min}, \epsilon_j^{\max}]$ based on the experimental observations. In our experiment on a qubit system, the interval is typically too small to be visible and the errors can be approximately determined as $\epsilon_j \approx \epsilon_j^{\min} \approx \epsilon_j^{\max}$.

Table 1 | Parameters of the used qubit in “ScQ-P136”

Qubit index	1
Qubit frequency (GHz)	4.524
Readout frequency (GHz)	6.832
Anharmonicity (MHz)	0.294
Relaxation time, T1 (μ s)	22.88
Coherence time, T2 (μ s)	19.9

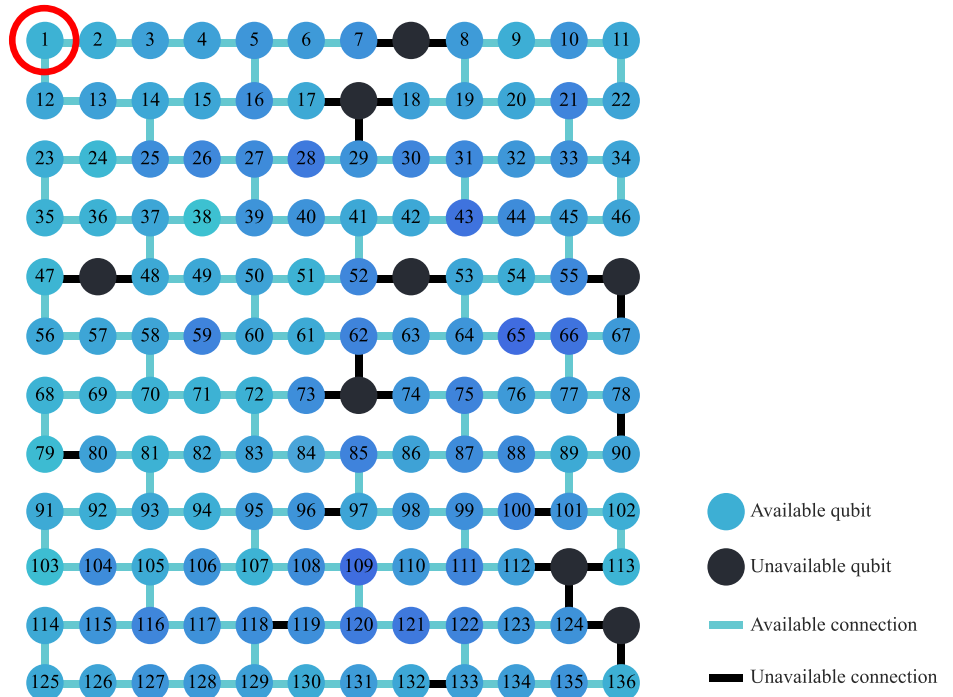
Table 2 | Parameters of the quantum processor “ScQ-P136”

T1 (μ s)	T2 (μ s)	Fidelity _{CNOT}	Fidelity _{Qubit}
Avg: 34.208	Avg: 17.485	Avg: 0.946	
min: 15.17	min: 0.95	min: 0.83	> 0.99
max: 59.1	max: 53.41	max: 0.996	

Characterization of the superconducting platform “ScQ-P136”

The experiments are performed on the “ScQ-P136” backend of the Quafu cloud quantum computing platform⁷¹. The parameters of the used qubit are shown in Table 1. The parameters and architecture of the processor can be found in Table 2 and Fig. 4.

Fig. 4 | Architecture of the quantum processor “ScQ-P136”. The used qubit is circled.



Data availability

All relevant source codes are available from the authors upon request.

Code availability

All relevant source codes are available from the authors upon request.

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References

- Heisenberg, W. über den anschaulichen inhalt der quantentheoretischen kinematik und mechanik. *Z. Phys.* **43**, 172 (1927).
- Robertson, H. P. The uncertainty principle. *Phys. Rev.* **34**, 163–164 (1929).
- Schrodinger, E. About Heisenberg uncertainty relation. *Sitzungsber. Preuss. Akad. Wiss. Berl. (Math. Phys.)* **19**, 296–303 (1930).
- Arthurs, E. & Kelly Jr, J. L. On the simultaneous measurement of a pair of conjugate observables. *Bell Syst. Tech. J.* **44**, 725–729 (1965).
- Arthurs, E. & Goodman, M. S. Quantum correlations: A generalized Heisenberg uncertainty relation. *Phys. Rev. Lett.* **60**, 2447–2449 (1988).
- Ozawa, M. Universally valid reformulation of the Heisenberg uncertainty principle on noise and disturbance in measurement. *Phys. Rev. A* **67**, 042105 (2003).
- Ozawa, M. Uncertainty relations for joint measurements of noncommuting observables. *Phys. Lett. A* **320**, 367–374 (2004).
- Ozawa, M. Uncertainty relations for noise and disturbance in generalized quantum measurements. *Ann. Phys.* **311**, 350–416 (2004).
- Ozawa, M. Physical content of Heisenberg's uncertainty relation: limitation and reformulation. *Phys. Lett. A* **318**, 21–29 (2003).
- Ozawa, M. Heisenberg's uncertainty relation: Violation and reformulation. *J. Phys.: Conf. Ser.* **504**, 012024 (2014).
- Hall, M. J. W. Prior information: How to circumvent the standard joint-measurement uncertainty relation. *Phys. Rev. A* **69**, 052113 (2004).
- Branciard, C. Error-tradeoff and error-disturbance relations for incompatible quantum measurements. *Proc. Natl Acad. Sci. USA* **110**, 6742 (2013).
- Branciard, C. Deriving tight error-trade-off relations for approximate joint measurements of incompatible quantum observables. *Phys. Rev. A* **89**, 022124 (2014).
- Lu, X.-M., Yu, S., Fujikawa, K. & Oh, C. H. Improved error-tradeoff and error-disturbance relations in terms of measurement error components. *Phys. Rev. A* **90**, 042113 (2014).
- Ozawa, M. Error-disturbance relations in mixed states. Eprint Arxiv arXiv:1404.3388 (2014).
- Buscemi, F., Hall, M. J. W., Ozawa, M. & Wilde, M. M. Noise and disturbance in quantum measurements: An information-theoretic approach. *Phys. Rev. Lett.* **112**, 050401 (2014).
- Busch, P., Lahti, P. & Werner, R. F. Colloquium: Quantum root-mean-square error and measurement uncertainty relations. *Rev. Mod. Phys.* **86**, 1261–1281 (2014).
- Busch, P., Heinonen, T. & Lahti, P. Heisenberg's uncertainty principle. *Phys. Rep.* **452**, 155–176 (2007).
- Qin, H.-H. et al. Uncertainties of genuinely incompatible triple measurements based on statistical distance. *Phys. Rev. A* **99**, 032107 (2019).
- Busch, P., Lahti, P. & Werner, R. F. Proof of Heisenberg's error-disturbance relation. *Phys. Rev. Lett.* **111**, 160405 (2013).
- Ma, W. et al. Experimental test of Heisenberg's measurement uncertainty relation based on statistical distances. *Phys. Rev. Lett.* **116**, 160405 (2016).
- Mao, Y.-L. et al. Testing Heisenberg-type measurement uncertainty relations of three observables. *Phys. Rev. Lett.* **131**, 150203 (2023).
- Hou, Z. et al. Minimal tradeoff and ultimate precision limit of multiparameter quantum magnetometry under the parallel scheme. *Phys. Rev. Lett.* **125**, 020501 (2020).
- Meng, X. et al. Machine learning assisted vector atomic magnetometry. *Nat. Commun.* **14**, 6105 (2023).
- Acín, A., Jané, E. & Vidal, G. Optimal estimation of quantum dynamics. *Phys. Rev. A* **64**, 050302 (2001).
- Taylor, M. A. & Bowen, W. P. Quantum metrology and its application in biology. *Phys. Rep.* **615**, 1–59 (2016).
- Albarelli, F., Barbieri, M., Genoni, M. & Gianani, I. A perspective on multiparameter quantum metrology: From theoretical tools to applications in quantum imaging. *Phys. Lett. A* **384**, 126311 (2020).
- Horodecki, R., Horodecki, P., Horodecki, M. & Horodecki, K. Quantum entanglement. *Rev. Mod. Phys.* **81**, 865–942 (2009).
- Nielsen, M. A. & Chuang, I. L. *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).
- CVX Research, I. CVX: Matlab software for disciplined convex programming, version 2.0. <http://cvxr.com/cvx> (2012).
- Diamond, S. & Boyd, S. CVXPY: A Python-embedded modeling language for convex optimization. *J. Mach. Learn. Res.* **17**, 1–5 (2016).
- Lofberg, J. Yalmip: a toolbox for modeling and optimization in matlab. In *2004 IEEE International Conference on Robotics and Automation (IEEE Cat. No. 04CH37508)*, 284–289 (2004).
- Cramér, H. *Mathematical Methods of Statistics* (Princeton University Press, Princeton, NJ, 1946).
- Fisher, R. A. On the mathematical foundations of theoretical statistics. *Philos. Trans. R. Soc. Lond. A* **222**, 309–368 (1922).
- Helstrom, C. W. *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
- Holevo, A. S. *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
- Genoni, M. G. et al. Optimal estimation of joint parameters in phase space. *Phys. Rev. A* **87**, 012107 (2013).
- Steinlechner, S. et al. Quantum-dense metrology. *Nat. Photonics* **7**, 626–630 (2013).
- Zhuang, Q., Zhang, Z. & Shapiro, J. H. Entanglement-enhanced lidars for simultaneous range and velocity measurements. *Phys. Rev. A* **96**, 040304 (2017).
- Huang, Z., Lupo, C. & Kok, P. Quantum-Limited Estimation of Range and Velocity. *PRX Quantum* **2**, 030303 (2021).
- Matsumoto, K. A geometrical approach to quantum estimation theory. arXiv **2111.09667** (2021). <https://doi.org/10.48550/arXiv.2111.09667>.
- Chen, H., Chen, Y. & Yuan, H. Incompatibility measures in multiparameter quantum estimation under hierarchical quantum measurements. *Phys. Rev. A* **105**, 062442 (2022).
- Chen, H., Chen, Y. & Yuan, H. Information geometry under hierarchical quantum measurement. *Phys. Rev. Lett.* **128**, 250502 (2022).
- Gill, R. D. & Massar, S. State estimation for large ensembles. *Phys. Rev. A* **61**, 042312 (2000).
- Zhu, H. & Hayashi, M. Universally fisher-symmetric informationally complete measurements. *Phys. Rev. Lett.* **120**, 030404 (2018).
- Lu, X.-M. & Wang, X. Incorporating Heisenberg's uncertainty principle into quantum multiparameter estimation. *Phys. Rev. Lett.* **126**, 120503 (2021).
- Suzuki, J. Explicit formula for the Holevo bound for two-parameter qubit-state estimation problem. *J. Math. Phys.* **57**, 042201 (2016).
- Sidhu, J. S., Ouyang, Y., Campbell, E. T. & Kok, P. Tight bounds on the simultaneous estimation of incompatible parameters. *Phys. Rev. X* **11**, 011028 (2021).
- Nagaoka, H. A new approach to cramer-rao bounds for quantum state estimation. In Hayashi, M. (ed.) *Asymptotic theory of quantum statistical inference: Selected Papers* (World Scientific, Singapore, 2005). Originally published as IEICE Technical Report, 89, 228, IT 89-42, 9-14 (1989).

50. Conlon, L. O., Suzuki, J., Lam, P. K. & Assad, S. M. Efficient computation of the Nagaoka-Hayashi bound for multiparameter estimation with separable measurements. *npj Quantum Inf.* **7**, 110 (2021).
51. Belliardo, F. & Giovannetti, V. Incompatibility in quantum parameter estimation. *N. J. Phys.* **23**, 063055 (2021).
52. Carollo, A., Spagnolo, B., Dubkov, A. A. & Valenti, D. On quantumness in multi-parameter quantum estimation. *J. Stat. Mech.: Theory Exp.* **2019**, 094010 (2019).
53. Ragy, S., Jarzyna, M. & Demkowicz-Dobrzański, R. Compatibility in multiparameter quantum metrology. *Phys. Rev. A* **94**, 052108 (2016).
54. Chen, Y. & Yuan, H. Maximal quantum fisher information matrix. *N. J. Phys.* **19**, 063023 (2017).
55. Liu, J., Yuan, H., Lu, X.-M. & Wang, X. Quantum fisher information matrix and multiparameter estimation. *J. Phys. A: Math. Theor.* **53**, 023001 (2020).
56. Chen, H. & Yuan, H. Optimal joint estimation of multiple rabi frequencies. *Phys. Rev. A* **99**, 032122 (2019).
57. Demkowicz-Dobrzański, R., Górecki, W. & Guţă, M. Multi-parameter estimation beyond quantum fisher information. *J. Phys. A: Math. Theor.* **53**, 363001 (2020).
58. Vidrighin, M. D. et al. Joint estimation of phase and phase diffusion for quantum metrology. *Nat. Commun.* **5**, 3532 (2014).
59. Crowley, P. J. D., Datta, A., Barbieri, M. & Walmsley, I. A. Tradeoff in simultaneous quantum-limited phase and loss estimation in interferometry. *Phys. Rev. A* **89**, 023845 (2014).
60. Yue, J.-D., Zhang, Y.-R. & Fan, H. Quantum-enhanced metrology for multiple phase estimation with noise. *Sci. Rep.* **4**, 5933 (2014).
61. Zhang, Y.-R. & Fan, H. Quantum metrological bounds for vector parameters. *Phys. Rev. A* **90**, 043818 (2014).
62. Liu, J. & Yuan, H. Control-enhanced multiparameter quantum estimation. *Phys. Rev. A* **96**, 042114 (2017).
63. Rocca, E. et al. Entangling measurements for multiparameter estimation with two qubits. *Quantum Sci. Technol.* **3**, 01LT01 (2017).
64. Razavian, S., Paris, M. G. A. & Genoni, M. G. On the quantumness of multiparameter estimation problems for qubit systems. *Entropy* **22** (2020). <https://www.mdpi.com/1099-4300/22/11/1197>.
65. Candeloro, A., Paris, M. G. A. & Genoni, M. G. On the properties of the asymptotic incompatibility measure in multiparameter quantum estimation. *J. Phys. A: Math. Theor.* **54**, 485301 (2021).
66. Yang, Y., Chiribella, G. & Hayashi, M. Attaining the ultimate precision limit in quantum state estimation. *Commun. Math. Phys.* **368**, 223–293 (2019).
67. Hou, Z. et al. "Super-Heisenberg" and Heisenberg scalings achieved simultaneously in the estimation of a rotating field. *Phys. Rev. Lett.* **126**, 070503 (2021).
68. Hou, Z. et al. Zero-trade-off multiparameter quantum estimation via simultaneously saturating multiple heisenberg uncertainty relations. *Science Advances* **7** (2021). <https://advances.sciencemag.org/content/7/1/eabd2986>.
69. Szczykulska, M., Baumgratz, T. & Datta, A. Multi-parameter quantum metrology. *Adv. Phys. X* **1**, 621 (2016).
70. Albarelli, F., Friel, J. F. & Datta, A. Evaluating the holevo cramér-rao bound for multiparameter quantum metrology. *Phys. Rev. Lett.* **123**, 200503 (2019).
71. Baqis quafu group. <https://quafu.baqis.ac.cn>.
72. BAQIS Quafu Group. Quafu-RL: The Cloud Quantum Computers based Quantum Reinforcement Learning. arXiv e-prints arXiv:2305.17966 (2023). 2305.17966.
73. BAQIS Quafu Group. Quafu-Qcover: Explore Combinatorial Optimization Problems on Cloud-based Quantum Computers. arXiv e-prints arXiv:2305.17979 (2023). 2305.17979.
74. Erhart, J. et al. Experimental demonstration of a universally valid error-disturbance uncertainty relation in spin measurements. *Nat. Phys.* **8**, 185–189 (2012).
75. Ringbauer, M. et al. Experimental joint quantum measurements with minimum uncertainty. *Phys. Rev. Lett.* **112**, 020401 (2014).

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Author contributions

H.Y. conceived the project. H.C. and H.Y. derived the theoretical results. H.C. and L.W. ran the Quafu experiment. H.C. and H.Y. wrote the manuscript. H.Y. supervised the project.

Competing interests

The authors declare no competing interests.

Additional information

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